Two Famous Concepts in $F$-Algebras

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Abstract

In this note, we extend "Gelfand- Mazur" theorem for a large class of $F$-algebras, for which the spectral and boundedness radius are finite and $A^*$ separates the points on $A$ and next, we generalize "Gleason, Kahane- Zelazko" theorem for a class of $F$-algebras, namely fundamental strongly sequential algebras which of course contain Banach algebras.

Keywords: Boundedness radius, Complete metrizable fundamental topological algebras, Nig-boundedness radius, Spectral radius, Strongly sequential algebras.

1 Introduction

In [10], Kinani, Oubbi and Oudadess show that in every unital locally convex algebra we have $\rho \leq \beta$ if and only if $\beta(x) < 1$ implies that $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$, where $\rho$ is the spectral radius and $\beta$ is the boundedness radius. Oubbi [12] extends the above result in the general setting. In 1990 [4], the first author introduces the concept of fundamental topological algebras which extend both locall convexity and locall boundedness. Anjidani in [3] proves that in every unital fundamental $F$-algebra, $\rho \leq \beta$.

We note that according to [1] every locally pseudoconvex (especially locally bounded and locally convex) topological algebra is fundamental topological algebra.

The concept of strongly sequential topological algebras has been mentioned by
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T. Husain [9]. Here, in section 3 we generalize "Gelfand- Mazur" theorem and in section 4 we extend "Gleason, Kahane- Zelazko" theorem. From now on, we have supposed all algebras are complex unital complete metrizable topological algebra.

2 Definitions and Related Results

In this section we have a collection of definitions and theorems which are all discussed in [3], [4], [5], [6], [7], [9], [10], [11] and [12].

Definition 2.1 Let $x$ be an element of a topological algebra $(A, \tau)$. We will say that $x$ is bounded if there exists some $r > 0$ such that the sequence $(\frac{x^n}{r^n})_n$ converges to zero. The radius of boundedness of $x$ with respect to $(A, \tau)$ is denoted by $\beta(x)$ and defined by

$$\beta(x) = \inf \{ r > 0 : (\frac{x^n}{r^n}) \to 0 \}$$

with the convention : $\inf \phi = +\infty$. We also say $A$ is a $\beta$ finite topological algebra if all elements of $A$ are bounded.

Definition 2.2 Let $A$ be a topological algebra. $A$ is said to be strongly sequential if there exists a neighbourhood $U$ of 0 such that for all $x \in U, x^n \to 0$ as $n \to \infty$.

Definition 2.3 A topological algebra $A$ is said to be fundamental if there exists $b > 1$ such that for every sequence $(a_n)$ of $A$, the convergence of $b^n(a_{n+1} - a_n)$ to zero in $A$ implies that $(a_n)$ is a Cauchy sequence.

Definition 2.4 Let $A$ be a topological algebra. We define

$$E(A) = \{ a \in A : \sum_{n=1}^{\infty} \frac{a^n}{n!} \text{ is convergent} \}.$$

Theorem 2.5 ([6]) Let $A$ be a topological algebra and $\{a, -a\} \subseteq E(A)$; then, $\exp(a)$ is invertible in $A$.

Theorem 2.6 ([3], [5]) Let $A$ be a complete metrizable fundamental topological algebra and $a \in A$. Then

$$\beta(a) < 1 \Rightarrow 1 - a \in \text{Inv}A.$$

Corollary 2.7 ([3]) Let $A$ be a complete metrizable fundamental topological algebra and $a \in A$. Then

$$\rho(a) \leq \beta(a).$$
Theorem 2.8 ([7], [10]) Let $A$ be a topological algebra. Then $\beta$ is continuous at zero if and only if $A$ is strongly sequential.

Definition 2.9 Let $A$ be a topological algebra. We say $A$ is $Q$–alge if $\text{Inv}A$ is open.

Theorem 2.10 ([11]) Let $A$ be a topological algebra. Then $\rho$ is continuous at zero if and only if $A$ is $Q$–alge.

Definition 2.11 Let $x$ be an element of a topological algebra $(A, \tau)$. We will say that $x$ is nig–bounded if there exists some $r > 0$ such that the sequence $\sum (\frac{x}{\lambda})^n$ converges in $(A, \tau)$ for every $\lambda \in \mathbb{C}$ with $|\lambda| > r$. The radius of nig-boundedness of $x$ is then defined as

$$\eta(x) = \inf\{r > 0 : \sum (\frac{x}{\lambda})^n \text{ converges in } (A, \tau) \forall \lambda \in \mathbb{C}, |\lambda| > r\} \quad (2)$$

Theorem 2.12 ([13]) Let $(A, \tau)$ be a topological algebra. Then the following equality holds:

$$\eta(x) = \max\{\rho(x), \beta(x)\} \quad (3)$$

3 Generalization of the Gelfand-Mazur Theorem

Anjidani in [3] extends Gelfand- Mazur theorem to the algebras that are fundamental $\beta$ finite and $A^*$ separates the points on $A$. We remember by corollary 2.7 that every fundamental $\beta$ finite topological algebra is also $\rho$ finite. We prove this theorem by similar proof as in [3] for topological algebras for which $\rho$ and $\beta$ are both finite and $A^*$ separates the points on $A$. This class of algebras obviously contains the fundamental $\beta$ finite topological algebras for which $A^*$ separates the points on $A$.

Theorem 3.1 Let $A$ be a topological algebra for which $\rho$ and $\beta$ are finite and $a \in A$. We have:

(i) The spectrum of $a$, is compact.

(ii) The mapping $F(z) = (z - a)^{-1}$ is holomorphic of $\mathbb{C} \setminus \text{Sp}(a)$ into $A$, that is, $\varphi \circ F$ is a holomorphic mapping of $\mathbb{C} \setminus \text{Sp}(a)$ into $\mathbb{C}$ for all $\varphi \in A^*$.

Proof: It is clear that $\text{Sp}(a)$ is bounded. Now we show $\text{Sp}(a)$ is closed. Let $\lambda \in \mathbb{C} \setminus \text{Sp}(a)$. We have

$$z - a = (\lambda - a)(1 - (\lambda - z)(\lambda - a)^{-1}), (z \in \mathbb{C}). \quad (4)$$

Since $\rho$ and $\beta$ are finite by theorem 2.12, $\eta$ is finite; in particular $\eta((\lambda - a)^{-1}) < \infty$. 

Let $z \in C$ with $|\lambda - z| < \frac{1}{\eta((\lambda - a)^{-1})}$. Then $\eta((\lambda - z)(\lambda - a)^{-1}) < 1$; by definition 2.11, $1 - (\lambda - z)(\lambda - a)^{-1}) \in InvA$ and
\[
(1 - (\lambda - z)(\lambda - a)^{-1}))^{-1} = 1 + \sum_{n=1}^{\infty} (\lambda - z)^n (\lambda - a)^{-n}.
\] (5)
Thus, if $|\lambda - z| < \frac{1}{\eta((\lambda - a)^{-1})}$, then by (4) and (5) we have
\[
(z - a)^{-1} = (\lambda - a)^{-1} + (\lambda - z)(\lambda - a)^{-2} + ...
\]
and so
\[
\varphi((z - a)^{-1}) = \varphi((\lambda - a)^{-1}) + (\lambda - z)\varphi((\lambda - a)^{-2}) + ...
\]
for every $\varphi \in A^*$. Hence, the mapping $F(z) = (z - a)^{-1}$ is holomorphic of $C \setminus Sp(a)$ into $A$.

Theorem 3.2 Let $A$ be a topological algebra that $\rho$ and $\beta$ are finite and $A^*$ separates the points on $A$. Then, the spectrum $Sp(a)$ of $a$ is nonempty for all $a \in A$.

Proof: Suppose $Sp(a) = \emptyset$. Let $\varphi \in A^*$ and choose $t$ with $t > \eta(a)$. We prove that the mapping $F(z) = \varphi((z - a)^{-1})$ is bounded on $E = \{z \in C : |z| \geq t\}$. Let $z \in E$; since $\eta(\frac{a}{t}) < 1$ by definition 2.11, $\sum_{n=1}^{\infty}(\frac{a}{t})^n$ is converges. We have
\[
|\varphi(z - a)^{-1}| \leq \frac{1}{|z|}(|\varphi(1)| + \sum_{n=1}^{\infty} |\varphi(a^n)| \frac{1}{t^n})
\] (8)
Thus, the mapping $F(z)$ is bounded on $E$ and hence it is bounded on $C$. Also, this mapping is an entire function. Therefore, by liouville’s theorem, it is a constant function. In particular, $\varphi((a^{-1})) = \varphi((1 - a)^{-1}))$. Since this is true for all $\varphi \in A^*$ and $A^*$ separates the points on $A$, we have $1 - a = -a$, which is a contradiction.

Corollary 3.3 (A generalization of Gelfand- Mazur theorem) Let $A$ be a topological algebra such that $\rho$ and $\beta$ are finite and $A^*$ separates the points on $A$. If $A$ is a division algebra, then $A$ is isomorphic to $C$.

Proof: It is clear.

4 Generalization of the Gleason, Kahane- Zelazko Theorem

Theorem 4.1 Let $A$ be a strongly sequential topological algebra that $\rho \leq \beta$. Then we have:
(i) $A$ is $Q$ – algebra.
(ii) Let $\varphi$ be a linear functional on $A$ such that $\varphi(1) = 1$ and $ker\varphi \subseteq singA$. Then $\varphi$ is continuous.
Proof: (i) Since $A$ is a strongly sequential topological algebra by theorem 2.8, $\beta$ is continuous at zero and since $\rho \leq \beta$ then $\rho$ is continuous at zero. By theorem 2.10 obtained result.

(ii) $\beta(x) < 1$ then $\varphi(x) \neq 1$, for if: $\varphi(x) = 1 = \varphi(1)$, then $1 - x \in \ker \varphi \subseteq \text{sing} A$; that by ([12] theorem 7) is contradiction.

If $|\varphi(x)| > 1$, take $x_0 = \frac{x}{\varphi(x)}$. Since $\beta(x_0) < 1$ then $\varphi(x_0) \neq 1$, that is impossible. Thus we must have, if $\beta(x) < 1$ then $|\varphi(x)| \leq 1$. Now we suppose $U$ be as in definition 2.2. We have $x \in \varepsilon U$ then $|\varphi(x)| < \varepsilon$.

Theorem 4.2 Let $A$ be a fundamental strongly sequential topological algebra. Then $E(A) = A$.

Proof: See ([2] proposition 4.1)

Theorem 4.3 (A generalization of the Gleason, Kahane-Zelazko theorem)
Let $A$ be a strongly sequential fundamental topological algebra. Then the following conditions are equivalent.

(i) $\varphi(1) = 1$ and $\ker \varphi \subseteq \text{sing}(A)$.

(ii) $\varphi(x) \in \text{Sp}(a)$ for all $a \in A$.

(iii) $\varphi : A \to \mathbb{C}$ is multiplicative.

Proof: (i) $\Leftrightarrow$ (ii) and (iii) $\Rightarrow$ (ii) can be proved easily and is omitted.

Now suppose (i) is true. By theorem (4.1(ii)) $\varphi$ is continuous. Suppose $a \in \frac{1}{2}U$ where $U$ is the neighbourhood satisfying in definition 2.2. Put $x = a^k$ for a natural $k \in \mathcal{N}$. We have $\beta(x) = \beta(a^k) = (\beta(a))^k < 1$, therefore $|\varphi(a^k)| \leq 1$.

Define $F : \mathcal{C} \to \mathcal{C}$ by

$$F(z) = \varphi(\exp(za)) = \varphi(1) + \sum_{n=1}^{\infty} \frac{z^n \varphi(a^n)}{n!}.$$  \hspace{1cm} (9)

Since $|\frac{z^n \varphi(a^n)}{n!}| \leq \frac{|z|^n}{n!}$, the series in (9) is converges and therefore $F$ is an entire function and $|F(z)| \leq 1 + \sum_{n=1}^{\infty} \frac{|z|^n}{n!} = \exp(|z|)$ for all $z \in \mathcal{C}$, so that $F$ has order at most 1. Since by theorem 2.5, $\exp(za) \in \text{Inv} A$ for all $z \in \mathcal{C}$, we have $F(z) \neq 0$. Therefore, there exists $\alpha \in \mathcal{C}$ such that:

$$F(z) = \exp(\alpha z) = 1 + \sum_{n=1}^{\infty} \frac{z^n (\alpha^n)}{n!} (z \in \mathcal{C}).$$

So, by comparing the two formulas for $F(z)$, we get $\varphi(a) = \alpha, \varphi(a^2) = \alpha^2$. Now let $x \in A$. There exists $r > 0$ such that $a = rx \in V$ and $\varphi(r^2 x^2) = (\varphi(rx))^2$ and so $\varphi(x^2) = (\varphi(x))^2$, i.e. $\varphi$ is a Jordan functional and hence by a famous algebraic proof ([8] proposition 16.6 ) it is multiplicative.
References


