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Two Famous Concepts in *F*-Algebras

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Abstract

In this note, we extend "Gelfand- Mazur" theorem for a large class of F-algebras, for which the spectral and boundedness radius are finite and A^* separates the points on A and next, we generalize " Gleason, Kahane- Zelazko" theorem for a class of F-algebras, namely fundamental strongly sequential algebras which of course contain Banach algebras.

Keywords: *Boundedness radius, Complete metrizable fundamental topological algebras, Nig-boundedness radius, Spectral radius, Strongly sequential algebras.*

1 Introduction

In [10], Kinani, Oubbi and Oudadess show that in every unital locally convex algebra we have $\rho \leq \beta$ if and only if $\beta(x) < 1$ implies that $(1-x)^{-1} = \Sigma_{n=0}^{\infty} x^n$, where ρ is the spectral radius and β is the boundedness radius. Oubbi [12] extends the above result in the general setting. In 1990 [4], the first author introduces the concept of fundamental topological algebras which extend both locall convexity and locall boundedness. Anjidani in [3] proves that in every unital fundamental F-algebra, $\rho \leq \beta$.
We note that according to [1] every locally pseudoconvex (especially locally bounded and locally convex) topological algebra is fundamental topological algebra.
The concept of strongly sequential topological algebras has been mentioned by

T.Husain [9]. Here, in section 3 we generalize "Gelfand- Mazur" theorem and in section 4 we extend " Gleason, Kahane- Zelazko" theorem. From now on, we have supposed all algebras are complex unital complete metrizable topological algebra.

2 Definitions and Related Results

In this section we have a collection of definitions and theorems which are all discussed in [3], [4], [5], [6], [7], [9], [10], [11] and [12].

Definition 2.1 *Let x be an element of a topological algebra (A, τ). We will say that x is bounded if there exists some $r > 0$ such that the sequence $(\frac{x^n}{r^n})_n$ converges to zero. The radius of boundedness of x with respect to (A, τ) is denoted by $\beta(x)$ and defined by*

$$\beta(x) = \inf\{r > 0 \ : \ (\frac{x^n}{r^n}) \to 0\} \tag{1}$$

with the convention : $\inf \phi = +\infty$. We also say A is a β finite topological algebra if all elements of A are bounded.

Definition 2.2 *Let A be a topological algebra.*
A is said to be strongly sequential if there exists a neighbourhood U of 0 such that for all $x \in U, x^n \to 0$ as $n \to \infty$.

Definition 2.3 *A topological algebra A is said to be fundamental if there exists $b > 1$ such that for every sequence (a_n) of A, the convergence of $b^n(a_{n+1} - a_n)$ to zero in A implies that (a_n) is a Cauchy sequence.*

Definition 2.4 *Let A be a topological algebra. We define*

$$E(A) = \{a \in A : \Sigma_{n=1}^{\infty} \frac{a^n}{n!} \text{ is convergent}\}.$$

Theorem 2.5 ([6]) *Let A be a topological algebra and $\{a, -a\} \subseteq E(A)$; then, exp(a) is invertible in A.*

Theorem 2.6 ([3], [5]) *Let A be a complete metrizable fundamental topological algebra and $a \in A$. Then*

$$\beta(a) < 1 \Rightarrow 1 - a \in InvA.$$

Corollary 2.7 ([3]) *Let A be a complete metrizable fundamental topological algebra and $a \in A$. Then*

$$\rho(a) \leq \beta(a).$$

Theorem 2.8 ([7], [10]) *Let A be a topological algebra. Then β is continuous at zero if and only if A is strongly sequential.*

Definition 2.9 *Let A be a topological algebra. We say A is $Q-algera$ if $InvA$ is open.*

Theorem 2.10 ([11]) *Let A be a topological algebra. Then ρ is continuous at zero if and only if A is $Q-algera$.*

Definition 2.11 *Let x be an element of a topological algebra (A,τ). We will say that x is nig-bounded if there exists some $r>0$ such that the sequence $\sum(\frac{x}{\lambda})^n$ converges in (A,τ) for every $\lambda \in \mathcal{C}$ with $|\lambda|>r$. The radius of nig-boundedness of x is then defined as*

$$\eta(x)=\inf\{r>0\ :\ \sum(\frac{x}{\lambda})^n \ converges\ in\ (A,\tau)\ \forall\lambda\in\mathcal{C}, |\lambda|>r\} \tag{2}$$

Theorem 2.12 ([13]) *Let (A,τ) be a topological algebra. Then the following equality holds:*

$$\eta(x)=\max\{\rho(x),\beta(x)\} \tag{3}$$

3 Generalization of the Gelfand-Mazur Theorem

Anjidani in [3] extends Gelfand- Mazur theorem to the algebras that are fundamental β finite and A^* separates the points on A. We remember by corollary 2.7 that every fundamental β finite topological algebra is also ρ finite. We prove this theorem by similar proof as in [3] for topological algebras for which ρ and β are both finite and A^* separates the points on A. This class of algebras obviously contains the fundamental β finite topological algebras for which A^* separates the points on A.

Theorem 3.1 *Let A be a topological algebra for which ρ and β are finite and $a\in A$. We have:*
(i) The spectrum of a, is compact.
(ii) The mapping $F(z)=(z-a)^{-1}$ is holomorphic of $\mathcal{C}\setminus Sp(a)$ into A, that is, φoF is a holomorphic mapping of $\mathcal{C}\setminus Sp(a)$ into $\mathcal{C}$ for all $\varphi\in A^*$.

Proof: It is clear that $Sp(a)$ is bounded. Now we show $Sp(a)$ is closed. Let $\lambda\in\mathcal{C}\setminus Sp(a)$. We have

$$z-a=(\lambda-a)(1-(\lambda-z)(\lambda-a)^{-1})),(z\in\mathcal{C}). \tag{4}$$

Since ρ and β are finite by theorem 2.12, η is finite; in particular $\eta((\lambda-a)^{-1})<\infty$.

Let $z \in \mathcal{C}$ with $|\lambda - z| < \frac{1}{\eta((\lambda-a)^{-1})}$. Then $\eta((\lambda - z)(\lambda - a)^{-1}) < 1$; by definition 2.11, $1 - (\lambda - z)(\lambda - a)^{-1}) \in InvA$ and

$$(1 - (\lambda - z)(\lambda - a)^{-1}))^{-1} = 1 + \Sigma_{n=1}^{\infty} (\lambda - z)^n (\lambda - a)^{-n}. \quad (5)$$

Thus, if $|\lambda - z| < \frac{1}{\eta((\lambda-a)^{-1})}$, then by (4) and (5) we have

$$(z - a)^{-1} = (\lambda - a)^{-1} + (\lambda - z)(\lambda - a)^{-2} + ... \quad (6)$$

and so

$$\varphi((z - a)^{-1}) = \varphi((\lambda - a)^{-1}) + (\lambda - z)\varphi((\lambda - a)^{-2}) + ... \quad (7)$$

for every $\varphi \in A^*$. Hence, the mapping $F(z) = (z - a)^{-1}$ is holomorphic of $\mathcal{C} \setminus Sp(a)$ into A.

Theorem 3.2 *Let A be a topological algebra that ρ and β are finite and A^* separates the points on A. Then, the spectrum $Sp(a)$ of a is nonempty for all $a \in A$.*

Proof: Suppose $Sp(a) = \emptyset$. Let $\varphi \in A^*$ and choose t with $t > \eta(a)$. We prove that the mapping $F(z) = \varphi((z-a)^{-1})$ is bounded on $E = \{z \in \mathcal{C} : |z| \geq t\}$. Let $z \in E$; since $\eta(\frac{a}{t}) < 1$ by definition 2.11, $\Sigma_{n=1}^{\infty}(\frac{a}{t})^n$ is converges. We have

$$|\varphi(z - a)^{-1}| \leq \frac{1}{|z|}(|\varphi(1)| + \Sigma_{n=1}^{\infty} \frac{|\varphi(a^n)|}{t^n}) \quad (8)$$

Thus, the mapping $F(z)$ is bounded on E and hence it is bounded on $\mathcal{C}$. Also, this mapping is an entire function. Therefore, by liouville's theorem, it is a constant function. In particular, $\varphi((-a^{-1})) = \varphi((1 - a)^{-1}))$. Since this is true for all $\varphi \in A^*$ and A^* separates the points on A, we have $1 - a = -a$, which is a contradiction.

Corollary 3.3 *(A generalization of Gelfand- Mazur theorem) Let A be a topological algebra such that ρ and β are finite and A^* separates the points on A. If A is a division algebra, then A is isomorphic to $\mathcal{C}$.*

Proof: It is clear.

4 Generalization of the Gleason, Kahane- Zelazko Theorem

Theorem 4.1 *Let A be a strongly sequential topological algebra that $\rho \leq \beta$. Then we have:*
(i) A is $Q-algebra$.
(ii) Let φ be a linear functional on A such that $\varphi(1) = 1$ and $ker\varphi \subseteq singA$. Then φ is continuous.

Proof: (i) Since A is a strongly sequential topological algebra by theorem 2.8 β is continuous at zero and since $\rho \leq \beta$ then ρ is continuous at zero. By theorem 2.10 obtained result.
(ii) $\beta(x) < 1$ then $\varphi(x) \neq 1$, for if ; $\varphi(x) = 1 = \varphi(1)$, then $1 - x \in ker\varphi \subseteq singA$; that by ([12] theorem 7) is contradiction.
If $|\varphi(x)| > 1$, take $x_0 = \frac{x}{\varphi(x)}$. Since $\beta(x_0) < 1$ then $\varphi(x_0) \neq 1$, that is impossible. Thus we must have, if $\beta(x) < 1$ then $|\varphi(x)| \leq 1$. Now we suppose U be as in definition 2.2. We have $x \in \frac{\varepsilon}{2}U$ then $|\varphi(x)| < \varepsilon$.

Theorem 4.2 *Let A be a fundamental strongly sequential topological algebra. Then $E(A) = A$.*

Proof: See ([2] proposition 4.1)

Theorem 4.3 *(A generalization of the Gleason, Kahane- Zelazko theorem) Let A be a strongly sequential fundamental topological algebra. Then the following conditions are equivalent.*
(i) $\varphi(1) = 1$ and $ker\varphi \subseteq sing(A)$.
(ii) $\varphi(x) \in Sp(a)$ for all $a \in A$.
(iii) $\varphi : A \to \mathcal{C}$ is multiplicative.

Proof: $(i) \Leftrightarrow (ii)$ and $(iii) \Rightarrow (ii)$ can be proved easily and is omitted.
Now suppose (i) is true. By theorem (4.1(ii)) φ is continuous. Suppose $a \in \frac{1}{2}U$ where U is the neighbourhood satisfying in definition 2.2. Put $x = a^k$ for a natural $k \in \mathcal{N}$. We have $\beta(x) = \beta(a^k) = (\beta(a))^k < 1$, therefore $|\varphi(a^k)| \leq 1$.
Define $F : \mathcal{C} \to \mathcal{C}$ by

$$F(z) = \varphi(exp(za)) = \varphi(1) + \Sigma_{n=1}^{\infty} \frac{z^n \varphi(a^n)}{n!}. \tag{9}$$

Since $|\frac{z^n\varphi(a^n)}{n!}| \leq \frac{|z^n|}{n!}$, the series in (9) is converges and therefore F is an entire function and $|F(z)| \leq 1 + \Sigma_{n=1}^{\infty} \frac{|z^n|}{n!} = exp(|z|)\ for\ all\ z \in \mathcal{C}$,
so that F has order at most 1. Since by theorem 2.5, $exp(za) \in InvA$ for all $z \in \mathcal{C}$, we have $F(z) \neq 0$. Therefore, there exists $\alpha \in \mathcal{C}$ such that:

$$F(z) = exp(\alpha\ z) = 1 + \ \Sigma_{n=1}^{\infty} \frac{z^n\ (\alpha^n)}{n!} (z \in \mathcal{C}).$$

So, by comparing the two formulas for $F(z)$, we get $\varphi(a) = \alpha, \varphi(a^2) = \alpha^2$. Now let $x \in A$. There exists $r > 0$ such that $a = rx \in V$ and $\varphi(r^2\ x^2) = (\varphi(rx))^2$ and so $\varphi(x^2) = (\varphi(x))^2$, i.e. φ is a Jordan functional and hence by a famous algebraic proof ([8] proposition 16.6) it is multiplicative.

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