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An Extension in the Domain of an n-Norm Defined on the Space of p-Summable Sequences

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Abstract

In [6], we have already studied the space of p-summable sequences (*i.e.* l^p) *as an n-normed space by defining a new n-norm* $\overline{\overline{\|\cdot,\cdots,\cdot\|}}_p$ *on it. In this paper, we shall extend the domain of the definition of n-norm* $\overline{\overline{\|\cdot,\cdots,\cdot\|}}_p$ *from the space* l^p *to different vector subspaces of the vector space* l^∞ *containing* l^p*. Further, we shall discuss on their derived norms also.*

Keywords: l^p *space,* l^∞ *space, norms, n-norms, derived norms.*

1 Introduction

In [9], Gähler initialy introduced the theory of *2-norm*, defined on a linear space, while that of *n-norm* can be found in [3] and has been studied in many papers such as [1, 2, 5]. Research works on sequence spaces regarded as *n-normed space* can be found in [1, 4, 6, 7, 8].

Definition 1.1. *Let* $\mathbf{X}$ *be a vector space over* $\mathbb{K}(=\mathbb{R}$ *or* $\mathbb{C})$ *of dimension* $d \geq n(n \geq 2)$*. A non-negative real valued function* $\|.,\ldots,.\|$ *defined on* $\mathbf{X^n}$ *satisfying the four conditions:*

(N1) $\|x^1,x^2,\cdots,x^n\| = 0$ *if and only if* $x^1,x^2,\cdots,x^n$ *are linearly dependent;*

(N2) $\|x^1, x^2, \cdots, x^n\|$ *is invariant under the permutation of* $x^1, x^2, \cdots, x^n$;

(N3) $\|\alpha \cdot x^1, x^2, \cdots, x^n\| = |\alpha| \cdot \|x^1, x^2, \cdots, x^n\|$;

(N4) $\|x^1 + y, x^2, \cdots, x^n\| \leq \|x^1, x^2, \cdots, x^n\| + \|y, x^2, \cdots, x^n\|$;

for all $x^1, x^2, \cdots, x^n, y \in \mathbf{X}$ *and for all* $\alpha \in \mathbb{K}$, *is called an* ***n-norm on X***, *and the pair* $(\mathbf{X}, \|., \ldots, .\|)$ *is called an* ***n-normed space***.

Definition 1.2. *Two n-norms* $\|\cdot, \cdots, \cdot\|_1$ *and* $\|\cdot, \cdots, \cdot\|_2$ *defined on a linear space* $\mathbf{X}$ *are said to be* **equivalent** *if and only if* $\exists\, K_1, K_2 > 0$ *such that:*

$$K_1 \cdot \|x^1, x^2, \cdots, x^n\|_1 \leq \|x^1, x^2, \cdots, x^n\|_2 \leq K_2 \cdot \|x^1, x^2, \cdots, x^n\|_1$$

for all $x^1, x^2, \cdots, x^n \in \mathbf{X}$.

Definition 1.3. *Let* $(\mathbf{X}, \|\cdot, \cdots, \cdot\|)$ *is an n-normed space and* $\{e^1, \cdots, e^n\}$ *is a set of linearly independent vectors in* $\mathbf{X}$ *then both of the functions* $\|\cdot\|_\infty^d$ *and* $\|\cdot\|_q^d$ *define a norm on* $\mathbf{X}$ *(known as* **derived norm** *with respect to the set* $\{e^1, \cdots, e^n\}$*) and they are equivalent, where*

1. $\|x\|_\infty^d = max\left\{\|x, e^{t_1}, \ldots, e^{t_{n-1}}\| : \{t_1, \ldots, t_{n-1}\} \subset \{1, \ldots, n\}\right\}$

2. $\|x\|_q^d = \left(\sum_{\{t_1,\ldots,t_{n-1}\}\subset\{1,\ldots,n\}} \|x, e^{t_1}, \ldots, e^{t_{n-1}}\|^q\right)^{1/q}; \quad 1 \leq q < \infty.$

In this paper, we shall focus on l^p and l^∞, where

$$l^p = \left\{ x = (x_i)_{i=0}^\infty : \sum_{i=0}^\infty |x_i|^p < \infty \quad \text{where} \quad x_i \in \mathbb{K}, \quad \text{for all} \quad i = 0, 1, 2, \ldots \right\}$$

and

$$l^\infty = \left\{ x = (x_i)_{i=0}^\infty : \sup_{0 \leq i < \infty} |x_i| < \infty \right\}.$$

As we know that $(l^p, \|\cdot\|_p)$ is a *Banach space* where $\|x\|_p = \left\{\sum_{i=0}^\infty |x_i|^p\right\}^{1/p}$ as well as $(l^\infty, \|\cdot\|_\infty)$ also becomes a *Banach space* with norm $\|x\|_\infty = \sup_{0 \leq i < \infty} |x_i|$; while $(l^p, \|\cdot\|_\infty)$ forms simply a *normed space*.

In [6], for our convenience and need we have denoted the set of whole numbers as $\mathbb{N} = \{0, 1, 2, \ldots\}$, which is also considered as a sequence $\mathbb{N} = (0, 1, 2, \ldots)$. Further, we have denoted the sequence $\mathbb{N} = (0, 1, 2, \ldots)$ in the form of *n-consecutive terms notation* as

$$\mathbb{N} = (0, 1, 2, \ldots) = (nl, nl + 1, \ldots, nl + (n - 1))_{l=0}^\infty$$

and expressed as

$$\begin{aligned} \mathbb{N} \;=\; & (n \cdot 0 = 0, n \cdot 0 + 1 = 1, \ldots, n \cdot 0 + (n - 1) = n - 1, \\ & n \cdot 1 = n, n \cdot 1 + 1 = n + 1, \ldots, n \cdot 1 + (n - 1) = 2n - 1, \ldots) \end{aligned}$$

Let $\overline{\overline{\mathbb{N}}} = \left(\overline{\overline{m}}_{nk}, \overline{\overline{m}}_{nk+1}, \ldots, \overline{\overline{m}}_{nk+(n-1)}\right)_{k=0}^{\infty}$ be a rearrangement of the sequence $\mathbb{N}$. Then for any n vectors

$$x^t = \left(x^t_{nl}, x^t_{nl+1}, \ldots, x^t_{nl+(n-1)}\right)_{l=0}^{\infty} \in l^p\,; \quad t = 1, 2, \ldots, n$$

the n vectors

$$\overline{\overline{x}}^t = \left(x^t_{\overline{\overline{m}}_{nk}}, x^t_{\overline{\overline{m}}_{nk+1}}, \cdots, x^t_{\overline{\overline{m}}_{nk+(n-1)}}\right)_{k=0}^{\infty}\,; \quad t = 1, 2, \ldots, n$$

are called *parallel rearrangements* of $x^1, x^2, \cdots, x^n$ respectively.

In [6], we have observed that $\left(l^p, \overline{\overline{\|\cdot, \cdots, \cdot\|}}_p\right)$ is an *n-normed space*, but not complete where

$$\overline{\overline{\|x^1, x^2, \cdots, x^n\|}}_p = sup\{|\overline{\overline{x}}^1, \overline{\overline{x}}^2, \ldots, \overline{\overline{x}}^n| : \overline{\overline{x}}^1, \overline{\overline{x}}^2, \ldots, \overline{\overline{x}}^n \quad \text{are parallel rearrangements of } x^1, x^2, \cdots, x^n \quad \text{respectively}\}. \tag{1}$$

and

$$\left|\overline{\overline{x}}^1, \overline{\overline{x}}^2, \ldots, \overline{\overline{x}}^n\right| = \left(\sum_{k=0}^{\infty} \left| det \begin{pmatrix} x^1_{\overline{\overline{m}}_{nk}} & x^1_{\overline{\overline{m}}_{nk+1}} & \cdots & x^1_{\overline{\overline{m}}_{nk+(n-1)}} \\ x^2_{\overline{\overline{m}}_{nk}} & x^2_{\overline{\overline{m}}_{nk+1}} & \cdots & x^2_{\overline{\overline{m}}_{nk+(n-1)}} \\ \cdots & \cdots & \cdots & \cdots \\ x^n_{\overline{\overline{m}}_{nk}} & x^n_{\overline{\overline{m}}_{nk+1}} & \cdots & x^n_{\overline{\overline{m}}_{nk+(n-1)}} \end{pmatrix} \right|^p \right)^{1/p}. \tag{2}$$

Moreover, we have

$$\overline{\overline{\|x^1, x^2, \cdots, x^n\|}}_p \leq n!\|x^{\pi_1}\|_p \cdot \|x^{\pi_2}\|_{p/\infty} \cdots \|x^{\pi_n}\|_{p/\infty}; \tag{3}$$

where $\{\pi_1, \pi_2 \ldots, \pi_n\}$ is any permutation of $\{1, 2, \ldots, n\}$ and $\|x^{\pi_t}\|_{p/\infty}$ means either $\|x^{\pi_t}\|_p$ or $\|x^{\pi_t}\|_\infty$ is taken freely.

In [1], Malčeski investigated that the function

$$\|x^1, x^2, \cdots, x^n\|_\infty := \sup_{i_1, \ldots, i_n} \left| det \begin{pmatrix} x^1_{i_1} & x^1_{i_2} & \ldots & x^1_{i_n} \\ x^2_{i_1} & x^2_{i_2} & \ldots & x^2_{i_n} \\ \ldots & \ldots & \ldots & \ldots \\ x^n_{i_1} & x^n_{i_2} & \ldots & x^1_{i_n} \end{pmatrix} \right| \tag{4}$$

defines an *n-norm* on l^∞ , where $i_1, \ldots, i_n \in \mathbb{N}$. But l^p is a subspace of l^∞ therefore we can show that $\|\ldots, .\|_\infty$ forms an *n-norm* on l^p also.

In [6, 7], we have already proved that these two *n-norms* $\overline{\overline{\|., \ldots, .\|}}_p$ and $\|\ldots, .\|_\infty$ defined on l^p are non-equivalent. Where as their *derived norms* with respect to the linearly independent set $\{e^1, \cdots, e^n\}$ are equivalent and equivalent to $\|.\|_\infty$, where $e^t = (\delta^t_i)_{i=0}^{\infty}$. For details see [7].

2 Results

Here, our aim is to investigate the possibilities of extending the domain of definition of the *n-norm* $\overline{\overline{\|\cdot,\cdots,\cdot\|}}_p$ from the domain l^p to other sequence spaces containing l^p.

If we take any arbitrary $z \in l^\infty$ and define

$$l^p + [z] = \{x + \alpha z : x \in l^p \quad \text{and} \quad \alpha \in \mathbb{K}\}.$$

then obviously $l^p + [z]$ is a subspace of $(l^\infty, \|\cdot\|_\infty)$.

Theorem 2.1. *The function* $\overline{\overline{\|\cdot,\cdots,\cdot\|}}_p$ *defines an n-norm on* $l^p + [z]$ *, for every arbitrary* $z \in l^\infty$*. Moreover,*

$$\overline{\overline{\|x^1 + z^1, x^2 + z^2, \ldots, x^n + z^n\|}}_p \leq n!\|x^{\pi_1}\|_p \cdot \|x^{\pi_2} + z^{\pi_2}\|_\infty \cdots \|x^{\pi_n} + z^{\pi_n}\|_\infty + n!\|x^{\pi_2}\|_p \cdot \|x^{\pi_3} + z^{\pi_3}\|_\infty \cdots \|x^{\pi_n} + z^{\pi_n}\|_\infty \cdot \|z^{\pi_1}\|_\infty \tag{5}$$

for every scalar multiples $z^1, z^2, \ldots, z^n$ *of* z *and any permutation* $\{\pi_1, \pi_2 \ldots, \pi_n\}$ *of* $\{1, 2, \ldots, n\}$.

Proof: The proof is similar to proving that $\overline{\overline{\|\cdot,\cdots,\cdot\|}}_p$ defines an *n-norm* on l^p, as we have done in [6]. Here we are going to establish the inequality only. Let $x^1 + z^1, x^2 + z^2, \ldots, x^n + z^n \in l^p + [z]$ and $\overline{\overline{\mathbb{N}}} = \left(\overline{\overline{m}}_{nk}, \overline{\overline{m}}_{nk+1}, \ldots, \overline{\overline{m}}_{nk+(n-1)}\right)_{k=0}^\infty$ be a rearrangement of the sequence $\mathbb{N}$. By the properties of determinant we have

$$det \begin{pmatrix} x^1_{\overline{\overline{m}}_{nk}} + z^1_{\overline{\overline{m}}_{nk}} & x^1_{\overline{\overline{m}}_{nk+1}} + z^1_{\overline{\overline{m}}_{nk+1}} & \cdots & x^1_{\overline{\overline{m}}_{nk+(n-1)}} + z^1_{\overline{\overline{m}}_{nk+(n-1)}} \\ x^2_{\overline{\overline{m}}_{nk}} + z^2_{\overline{\overline{m}}_{nk}} & x^2_{\overline{\overline{m}}_{nk+1}} + z^2_{\overline{\overline{m}}_{nk+1}} & \cdots & x^2_{\overline{\overline{m}}_{nk+(n-1)}} + z^2_{\overline{\overline{m}}_{nk+(n-1)}} \\ \cdots & \cdots & \cdots & \cdots \\ x^n_{\overline{\overline{m}}_{nk}} + z^n_{\overline{\overline{m}}_{nk}} & x^n_{\overline{\overline{m}}_{nk+1}} + z^n_{\overline{\overline{m}}_{nk+1}} & \cdots & x^n_{\overline{\overline{m}}_{nk+(n-1)}} + z^n_{\overline{\overline{m}}_{nk+(n-1)}} \end{pmatrix}$$

$$= det \begin{pmatrix} x^1_{\overline{\overline{m}}_{nk}} & x^1_{\overline{\overline{m}}_{nk+1}} & \cdots & x^1_{\overline{\overline{m}}_{nk+(n-1)}} \\ x^2_{\overline{\overline{m}}_{nk}} + z^2_{\overline{\overline{m}}_{nk}} & x^2_{\overline{\overline{m}}_{nk+1}} + z^2_{\overline{\overline{m}}_{nk+1}} & \cdots & x^2_{\overline{\overline{m}}_{nk+(n-1)}} + z^2_{\overline{\overline{m}}_{nk+(n-1)}} \\ \cdots & \cdots & \cdots & \cdots \\ x^n_{\overline{\overline{m}}_{nk}} + z^n_{\overline{\overline{m}}_{nk}} & x^n_{\overline{\overline{m}}_{nk+1}} + z^n_{\overline{\overline{m}}_{nk+1}} & \cdots & x^n_{\overline{\overline{m}}_{nk+(n-1)}} + z^n_{\overline{\overline{m}}_{nk+(n-1)}} \end{pmatrix}$$

$$+ det \begin{pmatrix} z^1_{\overline{\overline{m}}_{nk}} & z^1_{\overline{\overline{m}}_{nk+1}} & \cdots & z^1_{\overline{\overline{m}}_{nk+(n-1)}} \\ x^2_{\overline{\overline{m}}_{nk}} + z^2_{\overline{\overline{m}}_{nk}} & x^2_{\overline{\overline{m}}_{nk+1}} + z^2_{\overline{\overline{m}}_{nk+1}} & \cdots & x^2_{\overline{\overline{m}}_{nk+(n-1)}} + z^2_{\overline{\overline{m}}_{nk+(n-1)}} \\ \cdots & \cdots & \cdots & \cdots \\ x^n_{\overline{\overline{m}}_{nk}} + z^n_{\overline{\overline{m}}_{nk}} & x^n_{\overline{\overline{m}}_{nk+1}} + z^n_{\overline{\overline{m}}_{nk+1}} & \cdots & x^n_{\overline{\overline{m}}_{nk+(n-1)}} + z^n_{\overline{\overline{m}}_{nk+(n-1)}} \end{pmatrix}.$$

As we know that the expansion of a determinant of order n consists of sum of $n!$ terms, among which each term is again a product of n terms, therefore breaking the last determinant along second row and using Minkowski inequality; (Keeping the fact in mind that z^1 , z^2 are linearly dependent.) by definition (2) we have

$$|\overline{\overline{x}}^1 + \overline{\overline{z}}^1, \overline{\overline{x}}^2 + \overline{\overline{z}}^2, \ldots, \overline{\overline{x}}^n + \overline{\overline{z}}^n| \leq n!\|x^1\|_p \cdot \|x^2 + z^2\|_\infty \cdots \|x^n + z^n\|_\infty + n!\|x^2\|_p \cdot \|x^2 + z^2\|_\infty \cdots \|x^n + z^n\|_\infty \cdot \|z^1\|_\infty.$$

Above is true for any rearrangement $\overline{\overline{\mathbb{N}}}$ of $\mathbb{N}$ and breaking determinant along any row, therefore

$$\overline{\overline{\|x^1 + z^1, x^2 + z^2, \ldots, x^n + z^n\|}}_p \leq n!\|x^{\pi_1}\|_p \cdot \|x^{\pi_2} + z^{\pi_2}\|_\infty \cdots \|x^{\pi_n} + z^{\pi_n}\|_\infty + n!\|x^{\pi_2}\|_p \cdot \|x^{\pi_3} + z^{\pi_3}\|_\infty \cdots \|x^{\pi_n} + z^{\pi_n}\|_\infty \cdot \|z^{\pi_1}\|_\infty.$$

In general, if we take any $n-1$ arbitrary vectors $v^1, v^2, \ldots, v^{n-1} \in l^\infty$ and define

$$l^p + [v^1] + [v^2] + \cdots + [v^{n-1}] = \left\{x + \alpha_1 v^1 + \alpha_2 v^2 + \cdots + \alpha_{n-1} v^{n-1} : x \in l^p \text{ and } \alpha_j \in \mathbb{K}\, ; j = 1, 2, \ldots, n-1\right\}.$$

It is clear that, $l^p + [v^1] + [v^2] + \cdots + [v^{n-1}]$ is a subspace of $(l^\infty, \|\cdot\|_\infty)$.

Corollary 2.2. *The function* $\overline{\overline{\|\cdot, \cdots, \cdot\|}}_p$ *defines an n-norm on* $l^p + [v^1] + [v^2] + \cdots + [v^{n-1}]$ *for* $v^1, v^2, \ldots, v^{n-1} \in l^\infty$*. Moreover,*

$$\begin{aligned}\overline{\overline{\|x^1 + z^1, x^2 + z^2, \ldots, x^n + z^n\|}}_p \leq{}& n!\|x^{\pi_1}\|_p \cdot \|x^{\pi_2} + z^{\pi_2}\|_\infty \cdots \|x^{\pi_n} + z^{\pi_n}\|_\infty \\ &+ n!\|x^{\pi_2}\|_p \cdot \|x^{\pi_3} + z^{\pi_3}\|_\infty \cdots \|x^{\pi_n} + z^{\pi_n}\|_\infty \cdot \|z^{\pi_1}\|_\infty \\ &+ n!\|x^{\pi_3}\|_p \cdot \|x^{\pi_4} + z^{\pi_4}\|_\infty \cdots \|x^{\pi_n} + z^{\pi_n}\|_\infty \cdot \|z^{\pi_1}\|_\infty \cdot \|z^{\pi_2}\|_\infty \\ &+ \cdots + n!\|x^{\pi_n}\|_p \cdot \|z^{\pi_1}\|_\infty \cdots \|z^{\pi_{n-1}}\|_\infty \end{aligned} \tag{6}$$

where each of the vectors $z^1, z^2, \ldots, z^n$ *is linear combination of* $v^1, v^2, \ldots, v^{n-1}$.

Proof: The proof can be done similar to the proof of theorem 2.1 by breaking the determinant successively along every row.

Obviously, the function $\|\ldots, .\|_\infty$ defines an *n-norm* on $l^p + [v^1] + [v^2] + \cdots + [v^{n-1}]$ also. Since, the *n-norms* $\overline{\overline{\|\cdot, \cdots, \cdot\|}}_p$ and $\|\ldots, .\|_\infty$ are non-equivalent on l^p (for details see [7]), therefore they must be non-equivalent on $l^p + [v^1] + [v^2] + \cdots + [v^{n-1}]$ also. But, it is easy to show that their *derived norms* with respect to the linearly independent set $\{e^1, \ldots, e^n : e^t = (\delta_i^t)_{i=0}^\infty ; \quad t = 1, 2, \ldots, n\}$ are equivalent and equivalent to $\|\cdot\|_\infty$.

Under some conditions, the function $\overline{\overline{\|\cdot, \cdots, \cdot\|}}_p$ may define an n-norm on $l^p + [v^1] + [v^2] + \cdots + [v^n]$ for $v^1, v^2, \ldots, v^n \in l^\infty$, before discussing such conditions, let us consider the following theorem.

Theorem 2.3. *The function* $\overline{\overline{\|\cdot,\cdots,\cdot\|}}_p$ *need not be an n-norm on* $l^p+[v^1]+[v^2]+\cdots+[v^n]$ *for* $v^1,v^2,\ldots,v^n\in l^\infty$. *Moreover, the function* $\overline{\overline{\|\cdot,\cdots,\cdot\|}}_p$ *fails to be an n-norm on* l^∞.

Proof: Taking n vectors $v^t=(v_i^t)_{i=0}^\infty\in l^\infty$; $t=1,2,\ldots,n$ where

$$v_i^t=\begin{cases}1\,, & \text{if } i\equiv(t-1)(mod\,n)\ ;\\ 0\,, & \text{otherwise}\end{cases}$$

obviously, $\|v^t\|_\infty=1$ therefore $v^t\in l^\infty$ for every $t=1,2,\ldots,n$. But

$$\overline{\overline{\|v^1,\cdots,v^n\|}}_p=\infty.$$

Thus the function $\overline{\overline{\|\cdot,\cdots,\cdot\|}}_p$ need not be an n-norm on $l^p+[v^1]+[v^2]+\cdots+[v^n]$ for $v^1,v^2,\ldots,v^n\in l^\infty$. Moreover, the function $\overline{\overline{\|\cdot,\cdots,\cdot\|}}_p$ fails to be an n-norm on l^∞.

Now we shall investigate those circumstances under which the function $\overline{\overline{\|\cdot,\cdots,\cdot\|}}_p$ define an n-norm on $l^p+[v^1]+[v^2]+\cdots+[v^n]$ for $v^1,v^2,\ldots,v^n\in l^\infty$.

Lemma 2.4. *If* $\overline{\overline{\|x^1,x^2\|}}_p<\infty$ *for* $x^1,x^2\in l^\infty$ *then for every* $x^3,\ldots,x^n\in l^\infty$

$$\overline{\overline{\|x^1,x^2,\cdots,x^n\|}}_p<\infty\,.$$

Moreover,

$$\overline{\overline{\|x^1,x^2,\cdots,x^n\|}}_p\le\frac{n!}{2}\|x^3\|_\infty\cdots\|x^n\|_\infty\overline{\overline{\|x^1,x^2\|}}_p\,.$$

Proof: Let $\overline{\overline{\mathbb{N}}}=\left(\overline{\overline{m}}_{nk},\overline{\overline{m}}_{nk+1},\ldots,\overline{\overline{m}}_{nk+(n-1)}\right)_{k=0}^\infty$ be a rearrangement of the sequence $\mathbb{N}$. Obviously, expanding the determinant

$$det\begin{pmatrix}x^1_{\overline{\overline{m}}_{nk}} & x^1_{\overline{\overline{m}}_{nk+1}} & \cdots & x^1_{\overline{\overline{m}}_{nk+(n-1)}}\\ x^2_{\overline{\overline{m}}_{nk}} & x^2_{\overline{\overline{m}}_{nk+1}} & \cdots & x^2_{\overline{\overline{m}}_{nk+(n-1)}}\\ \cdots & \cdots & \cdots & \cdots\\ x^n_{\overline{\overline{m}}_{nk}} & x^n_{\overline{\overline{m}}_{nk+1}} & \cdots & x^n_{\overline{\overline{m}}_{nk+(n-1)}}\end{pmatrix}$$

in the form of the sum of the terms like

$$x^3_{\overline{\overline{m}}_{nk+j_3}}\cdot x^4_{\overline{\overline{m}}_{nk+j_4}}\cdots x^n_{\overline{\overline{m}}_{nk+j_n}}\cdot\begin{vmatrix}x^1_{\overline{\overline{m}}_{nk+j_1}} & x^1_{\overline{\overline{m}}_{nk+j_2}}\\ x^2_{\overline{\overline{m}}_{nk+j_1}} & x^2_{\overline{\overline{m}}_{nk+j_2}}\end{vmatrix}$$

and then using the Minkowski inequality equation (2) gives

$$\left|\overline{\overline{x}}^1,\overline{\overline{x}}^2,\ldots,\overline{\overline{x}}^n\right|=\frac{n!}{2}\|x^3\|_\infty\cdots\|x^n\|_\infty\overline{\overline{\|x^1,x^2\|}}_p\,.$$

Hence, we have the lemma.

Example: If we take $x^1, x^2 \in l^\infty$ as follows:

$$x^1 = \left(x_i^1\right)_{i=0}^\infty \quad \text{and} \quad x^2 = \left(x_i^2\right)_{i=0}^\infty$$

where

$$x_i^1 = \begin{cases} 1 & ; i = 0, 1 \\ \frac{1}{i^{1/p}} & ; i \geq 2 \end{cases}$$

and

$$x_i^2 = \begin{cases} 0 & ; i = 0 \\ 1 & ; i = 1, 2 \\ \frac{1}{(i-1)^{1/p}} & ; i \geq 3 \end{cases}$$

obviously, $x^1, x^2 \notin l^p$ but $x^1, x^2 \in l^\infty$. Whereas

$$\overline{\overline{\|x^1, x^2\|}}_p < \infty$$

for, let $\overline{\overline{\mathbb{N}}} = \left(\overline{\overline{m}}_{nk}, \overline{\overline{m}}_{nk+1}, \ldots, \overline{\overline{m}}_{nk+(n-1)}\right)_{k=0}^\infty$ be a rearrangement of the sequence $\mathbb{N}$ then for every $t \in \mathbb{N}$, we have

$$\left(\sum_{k=0}^{t} \left| det \begin{pmatrix} x^1_{\overline{\overline{m}}_{2k}} & x^1_{\overline{\overline{m}}_{2k+1}} \\ x^2_{\overline{\overline{m}}_{2k}} & x^2_{\overline{\overline{m}}_{2k+1}} \end{pmatrix} \right|^p \right)^{1/p} \leq \left(\sum_{k=0}^{t} \left| x^1_{\overline{\overline{m}}_{2k}} \cdot x^2_{\overline{\overline{m}}_{2k+1}} \right|^p \right)^{1/p} + \left(\sum_{k=0}^{t} \left| x^1_{\overline{\overline{m}}_{2k+1}} \cdot x^2_{\overline{\overline{m}}_{2k}} \right|^p \right)^{1/p}$$

but both

$$\sum_{k=0}^{t} \left| x^1_{\overline{\overline{m}}_{2k}} \cdot x^2_{\overline{\overline{m}}_{2k+1}} \right|^p \leq 2\left(1 + \sum_{i=1}^{\infty} \frac{1}{i^2}\right) \text{ and } \sum_{k=0}^{t} \left| x^1_{\overline{\overline{m}}_{2k+1}} \cdot x^2_{\overline{\overline{m}}_{2k}} \right|^p \leq 2\left(1 + \sum_{i=1}^{\infty} \frac{1}{i^2}\right)$$

therefore

$$\overline{\overline{\|x^1, x^2\|}}_p < \infty\,.$$

Lemma 2.5. *If for $x^1, x^2, \cdots, x^n \in l^\infty$,*

$$\overline{\overline{\|x^1, x^2, \cdots, x^n\|}}_p < \infty$$

then

$$\overline{\overline{\|z^1, z^2, \ldots, z^n\|}}_p < \infty$$

for each z^t ; $t = 1, 2, \ldots, n$ is linear combination of $x^1, x^2, \cdots, x^n$.

Theorem 2.6. *If $\overline{\overline{\|v^1, \cdots, v^n\|}}_p < \infty$ for $v^1, v^2, \ldots, v^n \in l^\infty$, then the function $\overline{\overline{\|\cdot, \cdots, \cdot\|}}_p$ defines an n-norm on $l^p + [v^1] + [v^2] + \cdots + [v^n]$.*

Proof: In view of lemma 2.5, the proof is similar to the proof of corollary 2.2.

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