Gen. Math. Notes, Vol. 29, No. 2, August 2015, pp.1-16

ISSN 2219-7184;

www.i-csrs.org

Available free online at http://www.geman.in

Vector Valued Orlicz Sequence Space Generilazed with an Infinite Matrix and Some of its Specific Characteristics

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(Received: 2-6-15 / Accepted: 24-7-15)

Abstract

The primary purpose of the present study is to present the vector valued sequence space $F(A, X_k, M, p, s)$ and to study the closed subspace of it. Where, F is a normal sequence algebra with absolutely monotone norm $\|.\|_F$ and having a Schauder base (e_k), where $e_k = (0, \cdots, 0, 1, 0, \cdots)$, with 1 in the $k-$th place; A is a nonnegative matrix; X_k is seminormed space over the complex field $\mathbb{C}$ with seminorm q_k for each $k \in \mathbb{N}$; M is an Orlicz function; $p = (p_k)$ be any sequence of strictly positive real numbers and s be any non-negative real number. We investigate important algebraic and topological characteristics of this space and also examine some inclusion relations on it. Our results are much more general than the corresponding results given by [23].

Keywords: *Orlicz function, Orlicz sequence space, Paranormed space, Vector valued sequence space.*

Note: *The main results of this paper were partially presented at the 1st International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2012) to be held on 03-27 September, 2012 in Prishtine, KOSOVO at the Prishtine University.*

1 Introduction

In recent years, the problems in Fourier series, power series and systems of equations having many variables have resulted in the first attemps to find out a theory of sequence spaces and infinite matrices. The theory of sequence spaces has also many other applications related to several other branches of functional analysis including theory of functions, summability theory and the theory of locally convex spaces of many problems concerning sequence spaces, the typical one is the inclusion problem (Abelian Theorems). That is, when spaces λ and μ are given, try to find out whether μ contains λ or not. There are many ways to introduce a sequence and a sequence space, but here we have preferred to give the definition as: Let w we denote the space of all real-valued sequences $x = (x_k)_{k=0}^{\infty}$. Any vector subspace of w is called a sequence space. As usual, we write c_0, c and l_∞ denote the sets of sequences that are convergent to zero, convergent and bounded, respectively. Also by l_1 and l_p; we denote the spaces of absolutely and $p-$ absolutely convergent series, respectively; where $1 < p < \infty$. We write e and $e^{(n)}$ $(n = 0, 1, \cdots)$ for the sequences with $e_k = 1$ $(k = 0, 1, \cdots)$ and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$. If $x \in w$ then $x^{[m]} = \sum\limits_{k=0}^{m} x_k e^{(k)}$ denotes the $m-$ section of x.

As well known, we call a sequence space X with a linear topology a $K-$space if and only if each of the maps $p_n : X \to \mathbb{R}$ defined by $p_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$. A $K-$space X is called an $FK-$space if and only if X is a complete linear metric space. On the other words; we can say that an $FK-$space is a complete total paranormed space. An $FK-$space whose topology is normable is called a $BK-$space, so a $BK-$space is a normed $FK-$space. The space ℓ_p $(1 \leq p < \infty)$ is a $BK-$space with $\|x\|_p = (\sum_k |x_k|^p)^{\frac{1}{p}}$ and c_0, c and ℓ_∞ are $BK-$space with $\|x\|_\infty = sup_k |x_k|$. An $FK-$space X is said to have $AK-$ property, if $\phi \subset X$ and $\{e^{(k)}\}$ is a basis for X, where e^k is a sequence whose only non-zero term is a 1 in $k-$th place for each $k \in \mathbb{N}$ and $\phi = span\{e^k\}$, the set of all finitely non-zero sequences. If ϕ is dense in X, then X is called an $AD-$space, thus AK implies AD. We know that the spaces c_0, cs and ℓ_p are $AK-$spaces, where $1 \leq p < \infty$.

A sequence $(b_n)_{n=0}^{\infty}$ in a linear metric space X is called Schauder basis if, for every $x \in X$, there exists a unique sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars such that $x = \sum\limits_{n=0}^{\infty} \lambda_n b_n$.

Orlicz sequence spaces are one of the generalizations of well-known sequence spaces ℓ_p, $p \geqslant 1$. They were examined by W. Orlicz in 1936. After that, the idea of Orlicz function M to construct the sequence space ℓ_M of all sequences of scalars (x_n) is used by J. Lindenstrauss and L. Tzafriri [21] such that

$$\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0.$$

The space ℓ_M equipped with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$$

is a $BK-$ space usually called an Orlicz sequence space.

The space ℓ_M becomes a Banach space which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = x^p$, $(1 \leqslant p < \infty)$. In the present note, we introduce and examine some properties of a sequence space defined by using Orlicz function M, which generalizes the well known Orlicz sequence space ℓ_M. Before introducing this sequence space, let us give some fundamental concepts contains essential definitions, results and terminological materials of which we shall make frequent use later.

An algebra X is defined as a linear space having an internal operation of multiplication of elements from X, provided that $xy \in X$, $x(yz) = (xy)z$, $x(y+z) = xy + xz$, $(x+y)z = xz + yz$ and $\lambda(xy) = (\lambda x)y = x(\lambda y)$, for a given scalar λ, and a normed algebra is defined as a normed linear space algebra where the inequality $\|xy\| \leqslant \|x\| \|y\|$ holds for all x, y; [22].

The scalar-valued sequence space F is called normal or solid if $y = (y_k) \in F$ whenever $|y_k| \leqslant |x_k|$, $k \in \mathbb{N}$, for some $x = (x_k) \in F$. Also F is called a sequence algebra if it is closed under the multiplication defined by $xy = (x_i y_i)$, $i \geq 1$. Should F is both normal and sequence algebra then it is called a normal sequence algebra. For example, c is a sequence algebra but not normal. w, ℓ_∞, c_0 and ℓ_p $(0 < p < \infty)$ are normal sequence algebras.

A norm $\|\cdot\|$ on a normal sequence space F is said to be absolutely monotone if $x = (x_k), y = (y_k) \in F$ and $|x_k| \leqslant |y_k|$ for all $k \in \mathbb{N}$ implies $\|x\| \leqslant \|y\|$, [19]. The norm

$$\|x\|_\infty = \sup |x_k|$$

over ℓ_∞, c, c_0 and the norm

$$\|x\| = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}$$

over ℓ_p for $p \geqslant 1$ are absolutely monotone.

We recall [20, 21] that an Orlicz function M is a function from $[0,\infty)$ to $[0,\infty)$ which is continuous, non-decreasing and convex with $M(0)=0$, $M(x)>0$ for all $x>0$ and $M(x)\to\infty$ as $x\to\infty$. Here we note that an Orlicz function is always unbounded and Orlicz function M can always be represented in the following integral form:

$$M(x)=\int_0^x p(t)\,dt,$$

where p, known as the kernel of M.

We easily obtain that M_1+M_2 and $M_1\circ M_2$ are Orlicz functions when M_1 and M_2 are Orlicz functions.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u if there exists a constant $K>0$ such that $M(2u)\leqslant KM(u)\,,\,u\geqslant 0$. It is easy to see always that $K>2$. The Δ_2-condition is equivalent to the inequality $M(\ell u)\leqslant K(\ell)M(u)$ which holds for all values of u and where l can be any number greater than unity [20].

We now introduce and examine the vector valued sequence space $F(A,X_k,$ $M,p,s)$.

Let $A=(a_{mk})$ be a nonnegative matrix, X_k be seminormed space over the complex field $\mathbb{C}$ with seminorm q_k for each $k\in\mathbb{N}$, and F be a normal sequence algebra with absolutely monotone norm $\|\cdot\|_F$ and having a Schauder basis (e_k), where $e_k=(0,\ldots,0,1,0,\ldots)$, with 1 in $k-$th place. Let $p=(p_k)$ be any sequence of strictly positive real numbers and s be any non-negative real number. Let (X_k,q_k) be an infinite sequence of seminormed spaces. Then we may construct the most general sequence spaces $s(X_k)$ such that $x=(x_k)\in s(X_k)$ if and only if $x_k\in X_k$ for each $k\in\mathbb{N}$. If we take $X_k=\mathbb{C}$ for each $k\in\mathbb{N}$, then we have w, the space of all complex-valued sequences. This case is called scalar-valued case. It is verification to show that $s(X_k)$ is a linear space over $\mathbb{C}$ under the coordinatewise operations. Let $x\in s(X_k)$ and $\lambda=(\lambda_k)$ be a scalar sequence such that $\lambda x=(\lambda_k x_k)$. We define for an Orlicz function M,

$$\begin{aligned} F(A,X_k,M,p,s)= \ & \Big\{x=(x_k)\in s(X_k): x_k\in X_k \text{ for each } k \text{ and } m\in\mathbb{N} \\ & \left(a_{mk}k^{-s}\left[M\left(\tfrac{q_k(x_k)}{\rho}\right)\right]^{p_k}\right)\in F \text{ for some } \rho>0\Big\}. \end{aligned}$$

The approach to construct a new sequence space has recently been employed. For instance, see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. For more detail certain sequence space, the reader may refer to Başar [2].

We now recall paranorm definition which will be used. The function g on X satisfies the properties of a paranorm

i) $g(\theta)=0$,

ii) $g(x) = g(-x)$

iii) $g(x+y) = g(x) + g(y)$

iv) $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$

for all $\alpha \in \mathbb{R}$ and all $x \in X$, where θ is the zero vector in the linear space X. Recall that a linear topological space X over the real field $\mathbb{R}$ with a paranorm obeying these rules (i-iv) is called a paranormed space.

A generalization of Minkowski inequality to normal sequence algebras having absolutely monotone seminorm has been introduced by [25] . Lemma 2.3 stating this extension is going to be used in order to put forward a topology of the space $F(A, X_k, M, p, s)$. For each m, we describe

$$g(x) = \inf\left\{\rho^{p_n/H} > 0\colon \left\|\left(a_{mk}k^{-s}\left[M\left(\frac{q_k\left(x_k\right)}{\rho}\right)\right]^{p_k}\right)\right\|_F^{1/H} \leqslant 1,\, m, n \in \mathbb{N}\right\}, \tag{1}$$

for $x = (x_k) \in F(A, X_k, M, p, s)$ where $H = \max\left(1, \sup p_k\right)$. It has been indicated that $F(A, X_k, M, p, s)$ is a complete paranormed space having the paranorm given by (1) provided that the seminormed space X_k is complete under the seminorm q_k for each $k \in \mathbb{N}$.

It is shown that if we can choose a suitable matrix A, sequence space F, the seminormed space X_k, the sequence of strictly positive real numbers (p_k), $s \geqslant 0$ and Orlicz function M, the space $F(A, X_k, M, p, s)$ results in the many number of known ordinary sequence spaces and as well as vector valued sequence spaces, as a particular case. For instance, let F be ℓ_1, $X_k = X$ (a vector space over $\mathbb{C}$) $q_k = q$ to be a seminorm on X in $F(A, X_k, M, p, s)$ and $a_{mk} = 1$ for all $m, k \in \mathbb{N}$ one gets the scalar valued sequence space $\ell_M\left(p, q, s\right)$ defined by Ç. A. Bektaş & Y. Altın [1].

If X_k is taken as a normed space, $a_{mk} = 1$ and $p_k = 1$ for all $m, k \in \mathbb{N}$ and $s = 0$, then the class $F(A, X_k, M, p, s)$ gives the class $F\left(X_k, M\right)$ defined by D. Ghosh & P. D. Srivastava [18]. Moreover, if $F = \ell_1$, $X_k = \mathbb{N}$, $a_{mk} = 1$ for all $m, k \in \mathbb{N}$ and $s = 0$ in $F(A, X_k, M, p, s)$, then one can easily derive the space $\ell_M(p)$ defined by S. D. Parashar & B. Choudhary [24]. Thus, the generalized sequence space $F(A, X_k, M, p, s)$ results in several spaces studied by several authors.

2 Linear Topological Structure of Vector Valued Orlich Sequence Space $F(A, X_k, M, p, s)$

In this subsection, we establish some algebraic and topological characteristics of vector valued Orlich sequence space with an infinite matrix $F\left(A, X_k, M, p, s\right)$

and investigate some inclusion relations on it. In order for argue the important characteristics of $F(A, X_k, M, p, s)$, we suppose that (p_k) is bounded. From now on, let's denote by h and C, the real numbers $\sup p_k$ and $\max\left(1, 2^{h-1}\right)$, respectively.

Let us begin with the theorem of one of our principal object of study.

Theorem 2.1 *$F(A, X_k, M, p, s)$ is a linear space over the complex field $\mathbb{C}$.*

Proof: Let's suppose that $x = (x_k),\ y = (y_k) \in F(A, X_k, M, p, s)$ and $\alpha, \beta \in \mathbb{C}$. Hence, there exist $\rho_1, \rho_2 > 0$ such that

$$\left(a_{mk}k^{-s}\left[M\left(\frac{q_k\left(x_k\right)}{\rho_1}\right)\right]^{p_k}\right), \left(a_{mk}k^{-s}\left[M\left(\frac{q_k\left(y_k\right)}{\rho_2}\right)\right]^{p_k}\right) \in F.$$

Let $\rho_3 = \max\left(2\left|\alpha\right|\rho_1, 2\left|\beta\right|\rho_2\right)$. Due to the fact that M is non-decreasing and convex, therefore, we can write

$$\begin{aligned} a_{mk}k^{-s}\left[M\left(\frac{q_k\left(\alpha x_k + \beta y_k\right)}{\rho_3}\right)\right]^{p_k} \leqslant C\Bigg\{ & a_{mk}k^{-s}\left[M\left(\frac{q_k\left(x_k\right)}{\rho_1}\right)\right]^{p_k} \\ & + a_{mk}k^{-s}\left[M\left(\frac{q_k\left(y_k\right)}{\rho_2}\right)\right]^{p_k}\Bigg\}. \end{aligned}$$

This newly derived inequality results in

$$\left(a_{mk}k^{-s}\left[M\left(\frac{q_k\left(\alpha x_k + \beta y_k\right)}{\rho_3}\right)\right]^{p_k}\right) \in F$$

Because F is a normal space. This clearly indicates $\alpha x + \beta y \in F(A, X_k, M, p, s)$. This is exactly what we want to prove.

Theorem 2.2 *$F(A, X_k, M, p, s)$ is a topological linear space, paranormed by g, defined by*

$$g(x) = \inf\left\{\rho^{p_n/H} > 0\colon \left\|\left(a_{mk}k^{-s}\left[M\left(\frac{q_k\left(x_k\right)}{\rho}\right)\right]^{p_k}\right)\right\|_F^{1/H} \leqslant 1,\ m, n \in \mathbb{N}\right\},$$

for each m, where $H = \max\left(1, h\right)$.

To prove this theorem we need the following lemma.

Lemma 2.3 *Let F be a normal sequence algebra, $\left\|\cdot\right\|_F$ be an absolutely monotone seminorm on F and let $p > 1$. Then*

$$\left\|(u+v)^p\right\|_F^{1/p} \leqslant \left\|u^p\right\|_F^{1/p} + \left\|v^p\right\|_F^{1/p},$$

for every $u = \left(u_n\right),\ v = \left(v_n\right) \in F$; [25].

Proof: (Proof of Theorem 2.2) We use a standard type procedure in proof of the theorem. Let's assume that $x=(x_k)$, $y=(y_k)\in F(A,X_k,M,p,s)$. The reader can obviously see that $g(x)=g(-x)$ and $g(\theta)=0$ for $\theta=(\theta_1,\theta_2,\ldots)$ the null element of $F(A,X_k,M,p,s)$ where θ_i is the zero element of X_i for each i.

Now, let us show the subadditivity of g. If we take $\alpha=\beta=1$ in Theorem 2.1, we can easily get

$$a_{mk}k^{-s}\left[M\left(\frac{q_k\left(x_k+y_k\right)}{\rho_3}\right)\right]^{p_k}\leqslant\left(a_{mk}k^{-s/H}\left[M\left(\frac{q_k\left(x_k\right)}{\rho_1}\right)\right]^{p_k/H}\right.$$
$$\left.+\,a_{mk}k^{-s/H}\left[M\left(\frac{q_k\left(y_k\right)}{\rho_2}\right)\right]^{p_k/H}\right)^{H}.$$

Lemma 2.3 allows us to write the following inequality

$$\left\|\left(a_{mk}k^{-s}\left[M\left(\frac{q_k\left(x_k+y_k\right)}{\rho_3}\right)\right]^{p_k}\right)\right\|_F^{1/H}\leqslant\left\|\left(a_{mk}k^{-s}\left[M\left(\frac{q_k\left(x_k\right)}{\rho_1}\right)\right]^{p_k}\right)\right\|_F^{1/H}$$
$$+\left\|\left(a_{mk}k^{-s}\left[M\left(\frac{q_k\left(y_k\right)}{\rho_2}\right)\right]^{p_k}\right)\right\|_F^{1/H}$$

in other words $g(x+y)\leqslant g(x)+g(y)$.

Now, let's show that the scalar multiplication is continuous. Let λ be any complex number. By (1), we get

$$g(\lambda x)=\inf\left\{\rho^{p_n/H}>0\colon\ \left\|\left(a_{mk}k^{-s}\left[M\left(\frac{q_k\left(\lambda x_k\right)}{\rho}\right)\right]^{p_k}\right)\right\|_F^{1/H}\leqslant 1,\ m,n\in\mathbb{N}\right\}.$$

Therefore

$$g(\lambda x)=\inf\left\{\left(|\lambda|\,r\right)^{p_n/H}>0\colon\ \left\|\left(a_{mk}k^{-s}\left[M\left(\frac{q_k\left(x_k\right)}{r}\right)\right]^{p_k}\right)\right\|_F^{1/H}\leqslant 1,\ m,n\in\mathbb{N}\right\},$$

where $r=\rho/\left|\lambda\right|$. For $|\lambda|^{p_n}\leqslant\max\left(1,|\lambda|^{\sup p_n}\right)$, we easily obtain

$$g(\lambda x)=\max\left(1,|\lambda|^{\sup p_n}\right)^{1/H}$$
$$.\inf\left\{r^{p_n/H}>0\colon\ \left\|\left(a_{mk}k^{-s}\left[M\left(\frac{q_k\left(x_k\right)}{r}\right)\right]^{p_k}\right)\right\|_F^{1/H}\leqslant 1,\ m,n\in\mathbb{N}\right\},$$

which converges to zero whenever x converges to zero in $F(A,X_k,M,p,s)$.

Let's assume that $\lambda_n\to 0$ and x is fixed in $F(A,X_k,M,p,s)$. Therefore,

$$t=(t_k)=\left(a_{mk}k^{-s}\left[M\left(\frac{q_k\left(x_k\right)}{\rho}\right)\right]^{p_k}\right)\in F$$

for each m, for some $\rho > 0$. For arbitrary $\varepsilon > 0$, let N be a positive integer such that

$$\left\| t - \sum_{k=1}^{N} t_k e_k \right\|_F = \left\| \sum_{k=N+1}^{\infty} t_k e_k \right\|_F < \left(\frac{\varepsilon}{2}\right)^H,$$

because (e_k) is a Schauder basis for F. Let $0 < |\lambda| < 1$, using convexity of M and absolutely monotonicity of $\|\cdot\|_F$ we easly obtain

$$\begin{aligned} \left\| \sum_{k=N+1}^{\infty} a_{mk} k^{-s} \left[M\left(\frac{q_k(\lambda x_k)}{\rho}\right)\right]^{p_k} e_k \right\|_F &\leqslant \left\| \sum_{k=N+1}^{\infty} a_{mk} k^{-s} \left[|\lambda| \, M\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k} e_k \right\|_F \\ &< \left(\frac{\varepsilon}{2}\right)^H. \end{aligned}$$

Due to the fact that M is continuous everywhere in $[0, \infty]$, it results in

$$f(u) =: \left\| \sum_{k=1}^{N} a_{mk} k^{-s} \left[M\left(\frac{q_k(u x_k)}{\rho}\right)\right]^{p_k} e_k \right\|_F$$

is continuous at 0. Thus there is $0 < \delta < 1$ such that $f(u) < (\varepsilon/2)^H$ for $0 < u < \delta$. Let K be a positive integer such that $|\lambda_n| < \delta$ for $n > K$, then for $n > K$

$$\left\| \sum_{k=1}^{N} a_{mk} k^{-s} \left[M\left(\frac{q_k(\lambda_n x_k)}{\rho}\right)\right]^{p_k} e_k \right\|_F^{1/H} < \frac{\varepsilon}{2}.$$

So

$$\left\| \left(a_{mk} k^{-s} \left[M\left(\frac{q_k(\lambda_n x_k)}{\rho}\right)\right]^{p_k} \right) \right\|_F^{1/H} < \frac{\varepsilon}{2}$$

for $n > K$, so that $g(\lambda x) \to 0$ as $\lambda \to 0$. This means that the scalar multiplication is continuous. As a conclusion, g is a paranorm on the space $F(A, X_k, M, p, s)$.

Remark 2.4 *If we take $F = \ell_1$, $a_{mk} = 1$ for all $m, k \in \mathbb{N}$, $(X_k, q_k) = (\mathbb{C}, |\cdot|)$, $p_k = 1$ for each $k \in \mathbb{N}$ and $s = 0$, then the paranorms defined on $F(A, X_k, M, p, s)$ and $\ell_M(p)$ will be same, and at the same time taking $a_{mk} = 1$ for all $m, k \in \mathbb{N}$, $q_k = \|\cdot\|_{X_k}$, $p_k = 1$ for each $k \in \mathbb{N}$ and $s = 0$ in (1), one can easily obtain the norm of $F(X_k, M)$.*

Theorem 2.5 *If X_k is complete under the seminorm q_k for each $k \in \mathbb{N}$, then $F(A, X_k, M, p, s)$ is complete with the paranorm (1).*

Proof: Let's suppose that (x^i) be any Cauchy sequence of the points lying in $F(A, X_k, M, p, s)$. We derived from (1) that

$$\left\| \left(a_{mk} k^{-s} \left[M \left(\frac{q_k\left(x_k^i - x_k^j\right)}{g\left(x^i - x^j\right)} \right) \right]^{p_k} \right) \right\|_F^{1/H} \leqslant 1.$$

Since F is a normal space and (e_k) is a Schauder basis of F, we obtain that

$$a_{mk} k^{-s} \left[M \left(\frac{q_k(x_k^i - x_k^j)}{g(x^i - x^j)} \right) \right]^{p_k} \|e_k\|_F \leqslant \left\| \left(a_{mk} k^{-s} \left[M \left(\frac{q_k(x_k^i - x_k^j)}{g(x^i - x^j)} \right) \right]^{p_k} \right) \right\|_F \leqslant 1.$$

We choose γ with $\gamma^H \|e_k\|_F > 1$ and $x_0 > 0$, such that

$$\gamma^H \|e_k\|_F \frac{x_0^H}{2} \left[p\left(\frac{x_0}{2}\right) \right]^{p_k} \geqslant 1,$$

where p is the kernel associated with M. So,

$$a_{mk} k^{-s} \left[M \left(\frac{q_k(x_k^i - x_k^j)}{g(x^i - x^j)} \right) \right]^{p_k} \|e_k\|_F \leqslant \gamma^H \|e_k\|_F \frac{x_0^H}{2} \left[p\left(\frac{x_0}{2}\right) \right]^{p_k}$$

for each $k \in \mathbb{N}$. Using the integral representation of Orlicz function M we have

$$a_{mk} k^{-s} \left[q_k(x_k^i - x_k^j) \right]^{p_k} \leqslant \gamma^H x_0^H \left[g(x^i - x^j) \right]^H. \tag{2}$$

For given $\varepsilon > 0$ we choose an integer i_0 such that

$$g(x^i - x^j) < \frac{\varepsilon^{1/H}}{\gamma x_0} \text{ for all } i, j > i_0. \tag{3}$$

From (2) and (3) we derive from

$$a_{mk} k^{-s} \left[q_k(x_k^i - x_k^j) \right]^{p_k} < \varepsilon \text{ for all } i, j > i_0$$

and so,

$$q_k(x_k^i - x_k^j) < \varepsilon \text{ for all } i, j > i_0.$$

Therefore, there exists a sequence $x = (x_k)$ such that $x_k \in X_k$ for each $k \in \mathbb{N}$ and

$$q_k(x_k^i - x_k) < \varepsilon \text{ as } i \to \infty,$$

for each fixed $k \in \mathbb{N}$. For given $\varepsilon > 0$, choose an integer $n > 1$ such that $g(x^i - x^j) < \varepsilon/2$, for all $i, j > n$ and a $\rho > 0$, such that $g(x^i - x^j) < \rho < \varepsilon/2$. Because F is a normal space and (e_k) is a Schauder basis of F,

$$\left\| \sum_{k=1}^{n} a_{mk} k^{-s} \left[M \left(\frac{q_k(x_k^i - x_k^j)}{\rho} \right) \right]^{p_k} e_k \right\|_F \leqslant \left\| \left(a_{mk} k^{-s} \left[M \left(\frac{q_k(x_k^i - x_k^j)}{\rho} \right) \right]^{p_k} \right) \right\|_F \leqslant 1.$$

For M is continuous, so by taking $j \to \infty$ and $i, j > n$ in the above inequality we easly obtain

$$\left\| \sum_{k=1}^{n} a_{mk} k^{-s} \left[M \left(\frac{q_k(x_k^i - x_k)}{2\rho} \right) \right]^{p_k} e_k \right\|_F < 1.$$

Letting $n \to \infty$, we have $g(x^i - x) < 2\rho < \varepsilon$ for all $i > n$. This means that (x^i) converges to x in the paranorm of $F(A, X_k, M, p, s)$. Now, if we prove that $x \in F(A, X_k, M, p, s)$, then the proof ends. Since $x^i = (x_k^i) \in F(A, X_k, M, p, s)$, there exists a $\rho > 0$ such that

$$\left(a_{mk} k^{-s} \left[M \left(\frac{q_k(x_k^i)}{\rho} \right) \right]^{p_k} \right) \in F.$$

Because of $q_k(x_k^i - x_k) \to 0$ as $i \to \infty$, for each fixed k we can choose a positive number δ_k^i satisfying $0 < \delta_k^i < 1$ such that

$$a_{mk} k^{-s} \left[M \left(\frac{q_k(x_k^i - x_k)}{\rho} \right) \right]^{p_k} < \delta_k^i a_{mk} k^{-s} \left[M \left(\frac{q_k(x_k^i)}{\rho} \right) \right]^{p_k}.$$

Consider

$$M \left(\frac{q_k(x_k)}{2\rho} \right) \leqslant M \left(\frac{q_k(x_k^i)}{\rho} \right) + M \left(\frac{q_k(x_k^i - x_k)}{\rho} \right).$$

In that case,

$$a_{mk} k^{-s} \left[M \left(\frac{q_k(x_k)}{2\rho} \right) \right]^{p_k} < C \left(1 + \delta_k^i \right) a_{mk} k^{-s} \left[M \left(\frac{q_k(x_k^i)}{\rho} \right) \right]^{p_k}.$$

This newly obtained formula results in

$$\left(a_{mk} k^{-s} \left[M \left(\frac{q_k(x_k)}{2\rho} \right) \right]^{p_k} \right) \in F,$$

because F is normal. That is, $x = (x_k) \in F(A, X_k, M, p, s)$. This in fact, concludes the proof.

Now, we examine some algebraic properties of the sequence spaces defined above and investigate some inclusion relations.

Theorem 2.6 *Let M and M_1 be two Orlicz functions. If M satisfies the Δ_2-condition, then*

$$F(A, X_k, M_1, p, s) \subseteq F(A, X_k, M \circ M_1, p, s).$$

Proof: To see that the inclusion in the theorem, let's assume that $x \in F(A, X_k, M_1, p, s)$. Therefore

$$\left(a_{mk}k^{-s}\left[M_1\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k}\right) \in F$$

for some $\rho > 0$. Since M and F satisfy the Δ_2-condition and normal, respectively, it is easy to see that following inequalities

$$\begin{aligned} a_{mk}k^{-s}\left[M\left(M_1\left(\frac{q_k(x_k)}{\rho}\right)\right)\right]^{p_k} &\leqslant a_{mk}k^{-s}\left[KM_1\left(\frac{q_k(x_k)}{\rho}\right)M(1)\right]^{p_k} \\ &\leqslant \max\left(1, [KM(1)]^h\right) a_{mk}k^{-s}\left[M_1\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k}. \end{aligned}$$

So, the previous computations show that $x \in F(A, X_k, M \circ M_1, p, s)$ and this completes the proof of the theorem.

Theorem 2.7 *Let M_1 and M_2 be two Orlicz functions. Then the following inclusions are hold for non-negative real numbers s_1, s_2, s:*

(i) $F(A, X_k, M_1, p, s) \cap F(A, X_k, M_2, p, s) \subseteq F(A, X_k, M_1 + M_2, p, s)$,

(ii) *If* $\limsup_{t\to\infty} M_1(t)/M_2(t) < \infty$, *then* $F(A, X_k, M_2, p, s) \subseteq F(A, X_k, M_1, p, s)$,

(iii) *If* $s_1 \leqslant s_2$, *then* $F(A, X_k, M_1, p, s_1) \subseteq F(A, X_k, M_1, p, s_2)$,

(iv) *If* $F_1 \subseteq F_2$, *then* $F_1(A, X_k, M_1, p, s) \subseteq F_2(A, X_k, M_1, p, s)$.

Proof: Since they are similar to each other, we will only prove part(i) and leave part (iii-iv) to reader.

(i) Let's suppose that $x \in F(A, X_k, M_1, p, s) \cap F(A, X_k, M_2, p, s)$. Therefore there exist some $\rho_1, \rho_2 > 0$ such that

$$\left(a_{mk}k^{-s}\left[M_1\left(\frac{q_k(x_k)}{\rho_1}\right)\right]^{p_k}\right), \left(a_{mk}k^{-s}\left[M_2\left(\frac{q_k(x_k)}{\rho_2}\right)\right]^{p_k}\right) \in F.$$

Letting $\rho = \max(\rho_1, \rho_2)$, we easly obtain

$$\begin{aligned} a_{mk}k^{-s}\left[(M_1 + M_2)\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k} \leqslant C\Bigg\{ & a_{mk}k^{-s}\left[M_1\left(\frac{q_k(x_k)}{\rho_1}\right)\right]^{p_k} \\ & + a_{mk}k^{-s}\left[M_2\left(\frac{q_k(x_k)}{\rho_2}\right)\right]^{p_k}\Bigg\}. \end{aligned}$$

Because of the fact that F is a normal space, $x \in F(A, X_k, M_1 + M_2, p, s)$.

(ii) It is easly find $K > 0$ such that $M_1(t)/M_2(t) \leqslant K$ for all $t \geqslant 0$, for $\limsup_{t\to\infty} M_1(t)/M_2(t) < \infty$. Let's suppose that $x \in F(A, X_k, M_2, p, s)$. Then, there exists a $\rho > 0$ such that

$$\frac{M_1\left(\frac{q_k(x_k)}{\rho}\right)}{M_2\left(\frac{q_k(x_k)}{\rho}\right)} \leqslant K.$$

So

$$a_{mk}k^{-s}\left[M_1\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k} \leqslant \max\left(1, K^h\right) a_{mk}k^{-s}\left[M_2\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k}.$$

Due to the fact that F is normal, $x \in F(A, X_k, M_1, p, s)$. This ends the proof.

Corollary 2.8 *We have*

(i) $F(A, X_k, p, s) \subseteq F(A, X_k, M, p, s)$ *for any Orlicz function* M *satisfying the* Δ_2-*condition,*

(ii) $F(A, X_k, M, p) \subseteq F(A, X_k, M, p, s)$ *for any Orlicz function* M.

3 A Closed Subspace of Vector Valued Orlicz Sequence Space $F(A, X_k, M, p, s)$

We describe $[F(A, X_k, M, p, s)]$ with the following

$$[F(A, X_k, M, p, s)] = \Big\{x = (x_k)\colon x_k \in X_k \text{ for each } k \in \mathbb{N} \text{ and } \ m \in \mathbb{N} \\ \left(a_{mk}k^{-s}\left[M\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k}\right) \in F \text{ for every } \rho > 0\Big\}.$$

Thus, the space $[F(A, X_k, M, p, s)]$ is obviously a subspace of $F(A, X_k, M, p, s)$, and its topology is introduced by the paranorm of $F(A, X_k, M, p, s)$ given by (1).

Theorem 3.1 $[F(A, X_k, M, p, s)]$ *is a complete paranormed space with the paranorm given by (1) if* (X_k, q_k) *is complete seminormed space for each* $k \in \mathbb{N}$.

Proof: Because $F(A, X_k, M, p, s)$ has just been a complete paranormed space under the paranorm (1) and $[F(A, X_k, M, p, s)]$ has been a subspace of $F(A, X_k, M, p, s)$, it is enough to show that it is closed. For this purpose, let us consider $(x^i) = ((x_k^i)) \in [F(A, X_k, M, p, s)]$ such that $g\left(x^i - x\right) \to 0$ as

$i \to \infty$, where $x = (x_k) \in F(A, X_k, M, p, s)$. Thus for a given $\xi > 0$, we can choose an integer i_0 such that

$$g\left(x^i - x\right) < \xi/2, \ \forall i > i_0.$$

Let's consider

$$\begin{aligned} a_{mk}k^{-s}\left[M\left(\frac{q_k(x_k)}{\xi}\right)\right]^{p_k} &\leqslant a_{mk}k^{-s}\left[\frac{1}{2}M\left(\frac{q_k(x_k^i - x_k)}{\xi/2}\right) + \frac{1}{2}M\left(\frac{q_k(x_k^i)}{\xi/2}\right)\right]^{p_k} \\ &\leqslant Ca_{mk}k^{-s}\left\{\left[M\left(\frac{q_k(x_k^i - x_k)}{g\left(x^i - x\right)}\right)\right]^{p_k} \right. \\ &\qquad \left. + \left[M\left(\frac{q_k(x_k^i)}{\xi/2}\right)\right]^{p_k}\right\}. \end{aligned}$$

Because

$$\left(a_{mk}k^{-s}\left[M\left(\frac{q_k(x_k^i - x_k)}{g\left(x^i - x\right)}\right)\right]^{p_k}\right), \left(a_{mk}k^{-s}\left[M\left(\frac{q_k(x_k^i)}{\xi/2}\right)\right]^{p_k}\right) \in F$$

and F is normal space,

$$\left(a_{mk}k^{-s}\left[M\left(\frac{q_k(x_k)}{\xi}\right)\right]^{p_k}\right) \in F.$$

This requires $x = (x_k) \in [F(A, X_k, M, p, s)]$ which clearly indicates that $[F(A, X_k, M, p, s)]$ is complete. In fact, this is exactly what we want to prove.

Theorem 3.2 $[F(A, X_k, M, p, s)]$ *is an AK-space.*

Proof: Let's suppose that $x = (x_k) \in [F(A, X_k, M, p, s)]$. Then,

$$\left(a_{mk}k^{-s}\left[M\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k}\right) \in F$$

for every $\rho > 0$. As (e_k) is a Schauder basis of F, for a given $\varepsilon \in (0,1)$, we are able to find out an arbitrary positive integer t_0 such that

$$\left\|\sum_{k=t_0}^{\infty} a_{mk}k^{-s}\left[M\left(\frac{q_k(x_k)}{\varepsilon}\right)\right]^{p_k} e_k\right\|_F < 1. \tag{4}$$

If we use the paranorm definition, then we obtain

$$g\left(x - x^{[t]}\right) = \inf\left\{\xi^{p_n/H} > 0 : \left\|\sum_{k=t+1}^{\infty} a_{mk}k^{-s}\left[M\left(\frac{q_k(x_k)}{\xi}\right)\right]^{p_k} e_k\right\|_F^{1/H} \leqslant 1, \ m, n \in \mathbb{N}\right\},$$

where $x^{[t]}$ represents the $t-$th section of x. Using this equality and (4), it is clear that

$$g\left(x - x^{[t]}\right) < \varepsilon \text{ for all } t > t_0.$$

Thus $[F(A, X_k, M, p, s)]$ is an AK-space.

Theorem 3.3 *Let $(x^i) = ((x_k^i))$ be a sequence of the elements of $[F(A, X_k, M, p, s)]$ and $x = (x_k) \in [F(A, X_k, M, p, s)]$. Then $x^i \to x$ in $[F(A, X_k, M, p, s)]$ if and only if*

(i) $x_k^i \to x_k$ *in* X_k *for each* $k \geqslant 1$,

(ii) $g(x^i) \to g(x)$ *as* $i \to \infty$.

Proof: The necessity part is clear.

Sufficiency. Let's assume that (i) and (ii) hold, and let t be an arbitrary positive integer. In this case

$$g(x^i - x) \leqslant g(x^i - x^{i[t]}) + g(x^{i[t]} - x^{[t]}) + g(x^{[t]} - x),$$

where $x^{i[t]}$, $x^{[t]}$ denote the $t-$th sections of x^i and x, respectively. Letting $i \to \infty$, we obtain

$$\begin{aligned}\limsup_{i\to\infty} g(x^i - x) &\leqslant \limsup_{i\to\infty} g(x^i - x^{i[t]}) + \limsup_{i\to\infty} g(x^{i[t]} - x^{[t]}) + g(x^{[t]} - x) \\ &\leqslant 2g(x^{[t]} - x).\end{aligned}$$

For t is arbitrary, letting $t \to \infty$, we get $\limsup_{i\to\infty} g\left(x^i - x\right) = 0$, i.e., $g\left(x^i - x\right) \to 0$ as $i \to \infty$.

Theorem 3.4 *If X_k is separable for each $k \in \mathbb{N}$, then $[F(A, X_k, M, p, s)]$ is separable.*

Proof: Let's assume that X_k is separable for each $k \in \mathbb{N}$. In that case, there exists a countable dense subset U_k of X_k. Let Z denotes the set of finite sequences $z = (z_k)$ where $z_k \in U_k$ for each $k \in \mathbb{N}$ and

$$(z_k) = (z_1, z_2, \ldots, z_t, \theta_{t+1}, \theta_{t+2}, \ldots)$$

for arbitrary $t \in \mathbb{N}$. Clearly, Z is a countable subset of $[F(A, X_k, M, p, s)]$. We shall prove that Z is dense in $[F(A, X_k, M, p, s)]$. Let's suppose that $x \in [F(A, X_k, M, p, s)]$. Because $[F(A, X_k, M, p, s)]$ is an AK-space, $g\left(x - x^{[t]}\right) \to 0$ as $t \to \infty$. So for a given $\varepsilon > 0$, there exists an integer $t_1 > 1$ such that

$$g\left(x - x^{[t]}\right) < \varepsilon/2 \text{ for all } t \geqslant t_1.$$

If $t = t_1$ is taken, then

$$g\left(x - x^{[t_1]}\right) < \varepsilon/2.$$

Let us choose $y = (y_k) = (y_1, y_2, \ldots, y_{t_1}, \theta_{t_1+1}, \theta_{t_1+2}, \ldots) \in Z$ such that

$$q_k\left(x_k^{[t_1]} - y_k\right) < \frac{\varepsilon}{2M(1)\, t_1 \left\|e_k\right\|_F} \text{ for each } k \in \mathbb{N}.$$

Now

$$g(x - y) \leqslant g\left(x - x^{[t_1]}\right) + g\left(x^{[t_1]} - y\right) < \varepsilon.$$

This requires that Z is dense in $[F(A, X_k, M, p, s)]$. Thus $[F(A, X_k, M, p, s)]$ is separable. This marks the end of the proof.

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