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Estimates of Two Partition Theoretic Functions

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Abstract

A partition say $\pi = (a_1, a_2, \cdots, a_k)$ of a positive integer n is said to be a modulo $-$ m partition of n if $a_i \equiv a_j (mod\ m)$ $\forall i, j$, where m is a positive integer greater than 1. Let $R_m(n, k)$ be the number of modulo-m partitions of n with exactly k parts. In this article, we show that:

$$R_m(n, 2) \sim \frac{n}{2m} \quad when \ \gcd(m, 2) = 1$$

and

$$R_m(n, 3) \sim \frac{n^2}{12m^2} \quad when \ \gcd(m, 3) = 1.$$

Estimate for $R_m(n, k)$ is conjectured.
Keywords: *Estimate, Restricted partitions, modulo-m partitions.*

1 Introduction

Finding estimate is an active research area in partition theory. The first estimate derived in partition theory is

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}, \tag{1}$$

where $p(n)$ is defined to be the number of partitions of n. Above estimate is an outcome of the exact formula given by Ramanujan [6] for $p(n)$. Subsequently, many authors have defined partition functions by imputing conditions over the parts and ensued to find its estimate. The estimate

$$p_A(n) \sim \frac{n^{k-1}}{(k-1)! \prod_{a\in A}} \tag{2}$$

was derived by several authors (see [2], [4], [7], [5], [9]) when A is a finite set of relatively prime positive integers, where $p_A(n)$ is defined to be the number of partitions of n with parts from the set A. When $A = H_{m,a} = \{n \in N : n \equiv a(mod\ m)\}$ with $0 \le a \le m-1$, then (see Theorem 6.4, [1])

$$p_{H_{m,a}}(n) \sim c_a n^{s_a} e^{\pi\sqrt{\frac{2n}{3m}}} \tag{3}$$

where

$$c_a = \Gamma\left(\frac{a}{m}\right) \pi^{\frac{a}{m}-1} 2^{-\frac{3}{2}-\frac{a}{2m}} 3^{-\frac{a}{2m}} m^{-\frac{1}{2}+\frac{a}{2m}}$$

and

$$s_a = -\frac{1}{2}\left(1+\frac{a}{m}\right).$$

In this article, we are concerned with a kind of partition which is associated with the function $p_{H_{m,a}}$. There are similar other types of partitions found in the literature (see [3], [8]) which arises as a result of considering congruence properties of the parts.

Definition 1.1 *Let n be a positive integer. By partition of n, we mean a sequence of non increasing positive integers say $\pi = (a_1, a_2, \cdots, a_k)$ such that $\sum_{i=1}^{k} a_i = n$. Partition π is said to be a $r-modulo-m$ partition of n if $a_i \equiv r(mod\ m)\ \forall i$, and $modulo-m$ partition if $a_i \equiv a_j(mod\ m)\ \forall i, j$, where m is a positive integer greater than 1 and $0 \le r \le m-1$. Let $R_{(m,r)}(n,k)$ and $R_m(n,k)$, respectively, be the number of $r-modulo-m$ partitions of n with exactly k parts and $modulo-m$ partitions of n with exactly k parts.*

Remark 1.2 *At this juncture, we remark that, if one defines $R_m(n)$ to be the number of modulo-m partitions of n, then*

$$R_m(n) = \sum_{a=0}^{m-1} p_{H_{a,m}}(n).$$

From whence one can get the following estimate:

$$R_m(n) \sim \sum_{a=0}^{m-1} c_a n^{s_a} e^{\pi\sqrt{\frac{2n}{3m}}}$$

where the constants c_a and s_a were as before.

Remark 1.3 *If* $\gcd(m,k) \geq 2$ *and* $(a_1, a_2, \cdots, a_k)$ *be a r- modulo-m partition of* n *with* k *parts, then from the congruence:*

$$n = a_1 + a_2 + \cdots + a_k \equiv rk(mod\ m),$$

(where r *is some positive integer less than* m*) it follows that,* n *is a multiple of* $\gcd(m,k) \geq 2$*. Thus, if* $\gcd(m,k) \geq 2$*, then numbers which are non multiples of* $\gcd(m,k)$ *cannot posses r- modulo-m partition with exactly* k *parts. So, we presumably take* $\gcd(m,k) = 1$*.*

The purpose of this article is to find the estimate of the restricted partition function $R_m(n,k)$ when $k = 2, 3$.

2 Main Results

It is well known that

$$p(n,k) \sim \frac{n^{k-1}}{k!(k-1)!}$$

where $p(n,k)$ is defined to be the number of partitions of n with exactly k parts.

Since $p(n,k) = p_{\{1,2,\cdots,k\}}(n-k)$, the above estimate follows as a special case of the estimate (2).

As main results of this paper, we give an asymptotic estimate for $R_m(n,2)$ and $R_m(n,3)$ which are respectively $\frac{1}{m}^{th}$ part of the estimate for $p(n,2)$ and $\frac{1}{m^2}^{th}$ part of the estimate for $p(n,3)$.

Following lemma is crucial for the main results.

Lemma 2.1 *We have*

$$R_{(m,r)}(n,k) = R_{(m,r)}(n-km,k) + R_{(m,r)}(n-r,k-1). \tag{4}$$

Proof: Let $\pi = (a_1, a_2, \cdots, a_k)$ be a r-modulo-m partition of n with exactly k parts. We enumerate π by considering the following cases.

Case(i) Let $a_k > r$. In this case, the mapping

$$(a_1, a_2, \cdots, a_k) \to (a_1 - m, a_2 - m, \cdots, a_k - m)$$

establishes an one-one correspondence between the following sets

- The set of all r-modulo-m partitions of n with exactly k parts and least part being greater than r.
- The set of all r-modulo-m partitions of $n - km$ with exactly k parts.

Note that, the cardinality of the latter set is $R_{(m,r)}(n-km,k)$.

Case(ii) Let $a_k = r$. In this case, the mapping

$$(a_1, a_2, \cdots, a_{k-1}, a_k) \to (a_1, a_2, \cdots, a_{k-1})$$

establishes an one to one correspondence between the following sets

- The set of all r-modulo-m partitions of n with exactly k parts and least part being equal to r.
- The set of all r-modulo-m partitions of $n-r$ wiht exactly $k-1$ parts.

Obviously, the cardinality of the latter set is $R_{(m,r)}(n-r,k-1)$.

Since the above two enumerations are mutually exclusive, the result follows.

Theorem 2.2 *Let $m > 1$ be an odd integer. Then we have*

$$R_m(n,2) \sim \frac{n}{2m}. \tag{5}$$

Proof: We see that

$$R_{(m,r)}(n,1) = \begin{cases} 1 \ \ if \ n \equiv r(mod\ m), \\ 0 \ otherwise \end{cases}$$

Thus from Division algorithm and lemma 2.1 it follows that

$$R_m(n,1) = 1 \ \forall n$$

Also, we have

$$R_{(m,r)}(n-r,1) = \begin{cases} 1 \ \ if \ n \equiv 2r(mod\ m), \\ 0 \ otherwise \end{cases}$$

It is well known that the congruence equation

$$2x \equiv n(mod\ m)$$

has solution if, and only if, $\gcd(m,2)$ divides n. In such case, there exist exactly $\gcd(m,2)$ solutions modulo m.

Here, since $\gcd(m,2) = 1$, there exist an unique r such that

$$R_{(m,r)}(n-r,1) = 1.$$

Consequently,

$$\sum_{r=0}^{m-1} R_{(m,r)}(n-r,1) = 1.$$

Then, from lemma 2.1, it follows that

$$R_m(n,2) - R_m(n-2m,2) = 1.$$

Consequently,

$$R_m(n,2) = \lfloor \frac{n}{2m} \rfloor.$$

Thus result follows.

Theorem 2.3 *Let $m > 1$ be a positive integer with $\gcd(m,3) = 1$. Then we have*

$$R_m(n,3) \sim \frac{n^2}{12m^2}$$

Proof: From lemma 2.1, it follows that

$$R_m(n,3) - R_m(n-3m,3) = \sum_{r=0}^{m-1} R_{(m,r)}(n-r,2).$$

Also from lemma 2.1, it follows that

$$R_{(m,r)}(n-r,2) = \sum_{k=0}^{\lfloor \frac{n-3r}{2m} \rfloor} R_{(m,r)}(n-2r-2km,1).$$

We see that

$$R_{(m,r)}(n-r-2km,1) = \begin{cases} 1 \;\; if \;\; n \equiv 3r (mod\ m), \\ 0 \; otherwise \end{cases}$$

Now, we notice that the congruence equation

$$3x \equiv n (mod\ m)$$

have unique solution modulo m if $\gcd(m,3) = 1$.

Here, since $\gcd(m,3) = 1$, there exist an unique $r^* \in \{0,1,\cdots,m-1\}$ satisfying $R_{(m,r^*)}(n-2r^*-2kmr^*,1) = 1$. Consequently,

$$R_m(n,3) - R_m(n-3m,3) = \lfloor \frac{n-3r^*}{2m} \rfloor + 1$$

Since $0 \le r^* \le m-1$, one can get the following inequality

$$\frac{n-3(m-1)}{2m} \le R_m(n,3) - R_m(n-3m,3) \le \frac{n}{2m} + 1$$

Take $n = (3m)!l + r$. Then application of the above inequality for $(3m-1)!$ times gives

$$\frac{1}{2m}\left(\sum_{i=0}^{(3m-1)!-1}(3m)!l - i3m - 3(m-1) + r\right)$$

$$\leq R_m((3m)!l + r, 3) - R_m((3m)!(l-1) + r, 3)$$

$$\leq \frac{1}{2m}\left(\sum_{i=0}^{(3m-1)!-1}((3m)!l - i3m + r + 2m)\right)$$

Equivalently,

$$\frac{1}{2m}((3m-1)!(3m)!l - \frac{(3m-1)!((3m-1)!-1)}{2}3m - 3(m-1)(3m-1)! + (3m-1)!r)$$

$$\leq R_m((3m)!l + r, 3) - R_m((3m)!(l-1) + r, 3)$$

$$\leq \frac{1}{2m}\left((3m-1)!(3m)!l - \frac{(3m-1)!((3m-1)!-1)}{2}3m + (3m-1)!r + (3m-1)!(2m)\right)$$

Summing the above inequality by taking $l = 1, 2, \cdots, s$ gives

$$\frac{1}{2m}\left((3m-1)!(3m)!\frac{s(s+1)}{2} - \frac{(3m-1)!((3m-1)!-1)}{2}3ms - 3(m-1)(3m-1)!s + (3m-1)!rs\right)$$

$$\leq R_m((3m)!s + r, 3) - R_m(r, 3)$$

$$\leq \frac{1}{2m}\left((3m-1)!(3m)!\frac{s(s+1)}{2} - \frac{(3m-1)!((3m-1)!-1)}{2}3ms + (3m-1)!rs + (3m-1)!(2m)s\right)$$

Dividing both sides by $((3m)!s + r)^2$ and letting $s \to \infty$, we get

$$\frac{1}{(4m)(3m)} \leq \lim_{s\to\infty} \frac{R_m((3m)!s + r, 3)}{((3m)!s + r)^2} \leq \frac{1}{(4m)(3m)}$$

Since above inequality is true for $r = 0, 1, \cdots, (3m)! - 1$, we get

$$\frac{1}{12m^2} \leq \lim_{n\to\infty} \frac{R_m(n, 3)}{n^2} \leq \frac{1}{12m^2}$$

Equivalently,

$$\lim_{n\to\infty} \frac{R_m(n, 3)}{n^2} = \frac{1}{12m^2}$$

Accordingly, we can write

$$R_m(n, 3) \sim \frac{n^2}{12m^2};$$

this is what we wish to prove.

2.1 An Open Problem

From the above two estimates, one can conjecture that

$$R_m(n,k) \sim \frac{n^{k-1}}{k!(k-1)!m^{k-1}}$$

when $\gcd(m,k) = 1$. Present method of derivation seems technically unsound to attack this general estimate.

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