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A Note on the Convergence of Random Riemann and Riemann-Stieltjes Sums to the Integral

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Abstract

Convergence in probability of random Riemann sums of a Lebesgue integrable function on $[0, 1)$ *to the integral has been proved. In this article we generalize the result to an abstract probability space under some natural conditions and we show* L_1*- convergence rather than convergence in probability..*

Keywords: *System of partitions, random Riemann Sums, Random Riemann-Stieltjes Sums, L1 Convergence, Integral.*

1 Introduction

The concept of random Riemann sums is considered in [1,2,3,4,5,6]. We present a somehow modified definition of it as follows.

Denote the interval $[0, 1)$ by I_0 and let I_0 be equipped with the Borel σ-algebra $\mathbf{B}$. Let m be the Lebesgue measure on $\mathbf{B}$.
Let $\mathcal{P}_0$ be a finite partition of I_0 by intervals of positive length. We note that any collection of disjoint non-empty intervals can be ordered naturally in terms of the natural order of real numbers. Let this order too be denoted by $<$. Let $J_i,\ 1 \leq i \leq n$, be the intervals constituting $\mathcal{P}_0$ s.t. $J_1 < J_2 < ... < J_n$. Corresponding to $\mathcal{P}_0$, there is a unique finite sequence $x_i,\ 0 \leq i \leq n$, of elements of I_0 s.t. $0 = x_0 < x_1 < ... < x_n = 1$ and s.t. x_{i-1} and x_i are the end points of J_i, for $1 \leq i \leq n$. In what follows $\mathcal{P}_0$ is fixed unless otherwise stated. For an arbitrary nonempty set S if $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions of S, we say $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$ and write $\mathcal{P}_1 \preceq \mathcal{P}_2$ if each element of $\mathcal{P}_1$ is a union of

elements of $\mathcal{P}_2$. In this article partitions of I_0 in the position that I_0 is assumed, i.e. when it is the range space of transformations, are as defined above. Hence if $\mathcal{P}_1$ and $\mathcal{P}_2$, are partitions of I_0, then $\mathcal{P}_1 \preceq \mathcal{P}_2$ if and only if the sequence corresponding to $\mathcal{P}_1$ is a subsequence of that corresponding to $\mathcal{P}_2$. The norm of $\mathcal{P}_0$ w.r.t. m is $\| \mathcal{P}_0 \| := \max\{m(J_i) \ : \ 1 \leq i \leq n\}$. For each $i, \ 1 \leq i \leq n$, let $t_i \in J_i$ be a random variable with uniform distribution in J_i.

Definition 1.1 *Let $f : I_0 \longrightarrow \mathbb{R}$ be a Lebesgue integrable function. For the partition $\mathcal{P}_0$ of I_0, the random Riemann sum of f w.r.t. $\mathcal{P}_0$ is defined to be the r.v.*

$$\mathcal{S}_{\mathcal{P}_0}(f) = \sum_{i=1}^{n} f(t_i)m(J_i).$$

Definition 1.2 *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space.Call for a partition $\mathcal{P}$ of Ω consisting of elements of $\mathcal{F}$, $\sup_{I\in\mathcal{P}} \mu(I)$ the norm of $\mathcal{P}$, w.r.t. μ and denote it by $| \mathcal{P} |_\mu$.*

Definition 1.3 *For a probability space $(\Omega, \mathcal{F}, \mu)$ a sequence $\{\Delta_n\}_{n\geq 1}$ of partitions of Ω is called a system of partitions if:*

1. *for each $n \geq 1$, Δ_n is a countable collection of elements of $\mathcal{F}$;*
2. *the collection $\bigcup_{n\geq 1} \Delta_n$ of subsets of Ω generates $\mathcal{F}$;*
3. $\lim_{n\longrightarrow\infty} | \Delta_n |_\mu = 0$.

Call a system of partitions decreasing if for each $n \geq 1, \ \Delta_{n+1}$ is a refinement of Δ_n.

Henceforth $\Delta_n, \ n \geq 1$, denotes a decreasing system of partitions of Ω and $\Delta = \cup_{n\geq 1}\Delta_n$.

It is not difficult to see that if for each $\omega, \ \{\omega\} \in \mathcal{F}$, the condition $| \Delta_n |_\mu \longrightarrow 0$ is equivalent to μ being diffuse, i.e. having no atoms.

Remark 1.4 *Euclidean spaces and more generally, locally compact second countable Hausdorff topological spaces and hence complete separable, i.e. Polish, metric spaces, with Borel σ-algebras and diffuse probability measures, when they admit such measures, yield decreasing systems of partitions.*

In the sequel we assume $(\Omega, \mathcal{F}, \mu)$ is a fixed probability space for which μ is atomless and there exists a decreasing system $\Delta_n, \ n \geq 1$ of partitions. Although to some stage we can proceed on a more general basis as described above, for the sake of simplicity and clarity we assume, in what follows, finite partitions for Ω instead of countable ones. Further we assume partitions have elements with positive, instead of nonnegative measures.

Let $\mathcal{P}$ be a (finite) partition of Ω consisting of measurable sets $A_1, A_2, ..., A_k$ (such that for each $i,\ 1 \leq i \leq k,\ \mu(A_i) > 0$).
For each $i,\ 1 \leq i \leq k$, let z_i be a random element of $A_i \in \mathcal{P}$, chosen according to the probability law $\mu_i(.) = \frac{\mu(.)}{\mu(A_i)}$.
There are randomization mechanisms, i.e. probability spaces which yield the required random elements. In all cases in this article, appropriate randomization mechanisms can be shown to exist[2].
Suppose $f : (\Omega, \mathcal{F}, \mu) \longrightarrow (\mathbb{R}, \mathbf{B}_{\mathbb{R}})$ is an integrable function where $\mathbf{B}_{\mathbb{R}}$ is the Borel σ-algebra in $\mathbb{R}$. In the sequel f will remain fixed.

Definition 1.5 *The random Riemann-Stieltjes sum of f w.r.t. $\mathcal{P}$ is defined to be*

$$S'_{\mathcal{P}}(f) = \sum_{1 \leq i \leq k} f(z_i)\mu(A_i).$$

Note that when $\Omega = I_0,\ \mathcal{F} = \mathbf{B},$ and $\mu = m$, then $S'_{\mathcal{P}}(f) = S_{\mathcal{P}}(f)$.

In [3] the sequence of random Riemann sums are considered for a fixed and non-random sequence of partitions $\{\Delta_n\}_{n\geq 1}$ of I_0 such that $\Delta_n \preceq \Delta_{n+1}, n \geq 1$, $\parallel \Delta_n \parallel \longrightarrow 0$ and t_i's are taken to be independent, and it is proved that such a sequence of random Riemann sums tends to $\int_{I_0} f dm$, a.s..

In [1] t_i's are taken to be independent and some results are proved for arbitrary but non-random, not necessarily being refined, sequences of partitions for which again the corresponding sequence of norms w.r.t. the Lebesgue measure m is assumed to tend to zero.
The following is a modified version of Proposition 2.1. of [1]:

For any $\epsilon > 0$, and any sequence of partitions $\mathcal{P}_n,\ n \geq 1$, of I_0 whose elements are finite unions of disjoint intervals, if $\lim_{n\longrightarrow\infty} \parallel \mathcal{P}_n \parallel = 0$, then

$$P(\mid S_{\mathcal{P}_n}(f) - \int_{I_0} f dm \mid > \epsilon) \longrightarrow 0.$$

In [6] based on the construction of a mapping from a general probability space $(\Omega, \mathcal{F}, \mu)$ to $(I_0, \mathbf{B}, m)$ under some reasonable and rather weak and general conditions, results of [3] and [1] are generalized, from the space $(I_0, \mathbf{B}, m)$ to $(\Omega, \mathcal{F}, \mu)$.
In this paper L_1-convergence of the random sequence $S'_{\Delta_n}(f),\ n \geq 1$, to $\int_\Omega f d\mu$ is proved. It will easily be seen (Theorem2, below) that by the method applied here, in the case of [1] too, without the assumption of independence of z_i's, L_1-convergence can be deduced.

2 Conclusion

These are the main results of the paper.

Theorem 2.1 *The random sequence $S'_{\Delta_n}(f), n \geq 1$, convergence to $\int_\Omega f d\mu$ in L_1.*

Theorem 2.2 *Let $g : I_0 \longrightarrow \mathbb{R}$ be a Lebesgue integrable function and $\{\mathcal{P}_n\}_{n\geq 1}$ be a sequence of partitions of I_0 s.t. $\| \mathcal{P}_n \| \longrightarrow 0$. Then the random Riemann sums of g, $S_{\mathcal{P}_n}(g)$, convergence to $\int_{I_0} gdm$ in L_1.*

Lemma 2.3 *Every indicator function can be approximated by a finite linear combination of indicator functions of elements of Δ, i.e. for each $A \in \mathcal{F}$ and each $\epsilon > 0$ there exists $M \in \mathbb{N}$ and members of Δ like $I_1, I_2, ..., I_M$ such that $E \mid \chi_A - \sum_{n=1}^{M} \chi_{I_n} \mid < \epsilon$*

Proof. For each $\epsilon > 0$, there is a sequence $I_n,\ n \geq 1$, of disjoint elements of Δ s.t. $A \subseteq \bigcup_{n\geq 1} I_n$, and

$$\mu(\bigcup_{n\geq 1} I_n) - \mu(A) < \frac{\epsilon}{2}.$$

Let N be s.t.

$$\mu(\bigcup_{n\geq 1} I_n) - \mu(\bigcup_{n=1}^{N} I_n) < \frac{\epsilon}{2}.$$

Then

$$\mu(\bigcup_{n=1}^{N} I_n - A) + \mu(A - \bigcup_{n=1}^{N} I_n) < \epsilon.$$

Hence

$$E \mid \chi_A - \Sigma_{n=1}^{N} \chi_{I_n} \mid < \epsilon. \blacksquare$$

Proof of Theorem 2.1: Let $\mathcal{L}$ be the set of all finite linear combinations of indicator functions of elements of Δ. We take the following standard steps.

1. It is clear that if a function g can be approximated by a member of $\mathcal{L}$, then so can cg for $c \in \mathbb{R}$.

2. If functions g_1 and g_2 can be approximated by members of $\mathcal{L}$, then so can $g_1 + g_2$.

3. If $g_1, g_2, ...$ is an increasing sequence of non-negative integrable functions increasing to the integrable function h, and for each $n \geq 1,\ g_n$ can be approximated by a member of $\mathcal{L}$, then so can h.

4. In view of the above steps, and Lemma1, for any $\epsilon > 0$ there is an element $g \in \mathcal{L}$ s.t. $E|f - g| < \epsilon$.

5. For each $I \in \Delta$, for sufficiently large n we have

$$S'_{\Delta_n}(\chi_I) = \mu(I),$$

hence for any $g \in \mathcal{L}$, for sufficiently large n,

$$S'_{\Delta_n}(g) - \int_\Omega g d\mu = 0.$$

It becomes clear that in terms of the L_1 norm, the set $\mathcal{L}$ is dense in the collection of integrable functions on $(\Omega, \mathcal{F}, \mu)$. Therefore for any $\epsilon > 0$, there exists $g \in \mathcal{L}$ s.t. $\int \mid f - g \mid < \epsilon$.
For any $n \geq 1$, we have

$$S'_{\Delta_n}(f) = S'_{\Delta_n}(g) + S'_{\Delta_n}(h),$$

and for any h and any n,

$$E \mid S'_{\Delta_n}(h) - \int_\Omega h d\mu \mid \leq E(S'_{\Delta_n}(|h|)) + \int_\Omega \mid h \mid = 2 \int_\Omega \mid h \mid .$$

Let $h = f - g$.
For sufficiently large n, we have

$$E|S'_{\Delta_n}(g) - \int_\Omega g d\mu| = 0.$$

So for sufficiently large n,

$$E \mid S'_{\Delta_n}(f) - \int_\Omega f d\mu \mid \leq E \mid S'_{\Delta_n}(g) - \int_\Omega g d\mu \mid + E \mid S'_{\Delta_n}(h) - \int_\Omega h d\mu \mid \ \leq 0 + 2\epsilon. \blacksquare$$

Proof of Theorem 2.2: Since the Riemann integrable functions are dense in L_1, in the step 5 above, it is sufficient to take g to be a Riemann integrable function s.t. $f = g + h$.$\blacksquare$

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