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Certain Characterizations for a Class of P-Valent Functions Defined by Salagean Differential Operator

Oyekan, Ezekiel Abiodun[1] and Ajai, Philip Terwase[2]

[1]Department of Mathematical Sciences
Ondo State University of Science and Technology (OSUSTECH),
P.M.B. 353 Okitipupa, Ondo State, Nigeria
E-mail: shalomfa@yahoo.com, ea.oyekan@osustech.edu.ng
[2]Department of Mathematics
Plateau State University, Bokkos, Nigeria
E-mail: philipajai2k2@yahoo.com

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Abstract

The aim of this paper is to give further properties of the class $S_p(A, B, b, \lambda)$ which was studied by Ajab et. al.[2]. In particular, we prove the extreme points, modified Hadamard products and inclusion properties for the class $S_p(A, B, b, \lambda)$.

Keywords: *P-valent function, Subordination, Salagean operator, extreme point, Hadamard product.*

1 Introduction

Salagean[4] defined the operator $D^{\lambda}f(z) = z + \sum_{k=2}^{\infty} k^{\lambda} a_k z^k$ where $\lambda \in N_0 = N \cup 0$.
Let S_p be the class of p-valent function which are analytic in the unit disk $U = \{z \in C : |z| < 1\}$ and can be written in the form $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, p \in N$.
In 2011, Ajal et. al.[2] defined the class $S_p(A, B, b, \lambda)$ as class of functions $f(z)$

satisfying the condition

$$1+\frac{1}{b}\left(\frac{z(D^{\lambda+p}f(z)^{'})}{D^{\lambda+p}f(z)}-p\right)\prec\frac{1+Az}{1+Bz} \tag{1}$$

where $\prec$ denote subordination, b is a non-zero complex number, A and B are thearbitrary constants with $-1\leq B<A\leq 1$. $D^{\lambda+p}$ is an extended Salagean operator defined by Eker and Seker [5] as

$$D^{\lambda+p}f(z)=D(D^{\lambda+p-1}f(z))=z^p+\sum_{k=1}^{\infty}(\frac{p+k}{p})^{\lambda}a_{p+k}z^{p+k} \tag{2}$$

where $\lambda\in N_0\cup 0$.

2 Preliminaries and Definitions

Let S be the class of analytic univalent functions $f(z)$ of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \tag{3}$$

that are defined in the open unit disk $U=\{z:|z|<1\}$.
Also let S_p denote the class of functions defined by

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}z^{p+k}, p\in N \tag{4}$$

which are analytic and p-valent in the unit disk $U=\{z:|z|<1\}$.
For $f(z)\in S$, Salagean in [4] introduced the operator:

$$D^0f(z)=f(z)$$

$$D^1f(z)=Df(z)=zf^{'}(z)$$

$$D^{\lambda}f(z)=D(D^{\lambda-1}f(z)),(\lambda\in N=\{1,2,3...\}).$$

We note that

$$D^{\lambda}f(z)=z+\sum_{k=2}^{\infty}k^{\lambda}a_kz^k,(\lambda\in N_0\cup 0).$$

Following Eker and Seker in [5], Ajab et. al.[2] gave the following inequalities for the functions $f(z)\in S_p$:

$$D^0f(z)=f(z)$$

$$D^1 f(z) = Df(z) = \frac{z}{p} f^{'}(z) = z^p + \sum_{k=1}^{\infty} (\frac{p+k}{p}) a_{p+k} z^{p+k}$$

$$D^{\lambda+p} f(z) = D(D^{\lambda+p-1} f(z)) = z^p + \sum_{k=1}^{\infty} (\frac{p+k}{p})^{\lambda} a_{p+k} z^{p+k}, \lambda \in N_0 \cup 0.$$

Definition 2.1. *[2] Let $S_p(A, B, b, \lambda)$ denote the subclass of S_p that consist of functions $f(z)$ which satisfy the condition*

$$1 + \frac{1}{b} \left(\frac{z(D^{\lambda+p} f(z)^{'})}{D^{\lambda+p} f(z)} - p \right) \prec \frac{1 + Az}{1 + Bz}$$

where $\prec$ denote subordination, b is a non-zero complex number, A and B are thearbitrary constants with $-1 \leq B < A \leq 1$. and $z \in U$.

This class is due to the class $M_p(A, B, b, n)$ defined by Ajab and Maslina[1] using Ruscheweyh derivatives.
By specializing the parameters for A, B, b, n *and* λ, the following subclasses studied by earlier authors are obtained
(i) $S_1(1, -1, b, 0) = C(b, 1)$ which was studied by Wiatrowski[3]
(ii) $S_1(A, B, b, 0) = C(A, B, b)$ which was studied by Ravichandran et.al.[6], (see [2])
Before we state and prove our main results we need the following definitions and theorem :

Theorem 2.2. *[4]*

$$\sum_{k=1}^{\infty} [k + |b(A - B) - Bk|] (\frac{p+k}{p})^{\lambda} |a_{p+k}| \leq |b|(A - B) \tag{5}$$

where $-1 \leq B < A \leq 1, \lambda \in N_0 \cup 0$ and $p \in N$.

Definition 2.3. *If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ are analytic in U, then their Hadamard product (or convolution), $f * g$ is the function defined by the power series*

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z) \tag{6}$$

The function $f * g$ is also analytic in U.

Definition 2.4. *Let $\tau(p)$ denote the subclass of S_p consistsing of functions of the form*

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, a_{p+k} \geq 0, p \in N \tag{7}$$

We denote $S_p^*(A,B,b,\lambda)$ the class obtained by taking the intersection of the class $S_p(A,B,b,\lambda)$ with the class $\tau(p)$. Thus we have

$$S_p^*(A,B,b,\lambda) = S_p(A,B,b,\lambda) \cap \tau(p) \tag{8}$$

3 Main Results

We begin by proving the following results.

3.1 Extreme Points

Theorem 3.1. *Let*

$$f_p(z) = z^p, p \in N \tag{9}$$

and

$$f_{p+k}(z) = z^p - \frac{|b|(A-B)}{[k+|b(A-B)-Bk|](\frac{p+k}{p})^\lambda} z^{p+k}, p,k \in N \tag{10}$$

Then $f(z) \in S_p(A,B,b,\lambda)$ iff it can be expressed in the form:

$$f(z) = \sum_{k=0}^{\infty} \delta_{p+k} f_{p+k}(z) \tag{11}$$

where

$$\delta_{p+k} \geq 0, \sum_{k=0}^{\infty} \delta_{p+k} = 1. \tag{12}$$

Proof. Let

$$f(z) = \sum_{k=0}^{\infty} \delta_{p+k} f_{p+k}(z) = z^p - \sum_{k=1}^{\infty} \frac{\delta_{p+k}|b|(A-B)}{[k+|b(A-B)-Bk|](\frac{p+k}{p})^\lambda} z^{p+k} \tag{13}$$

then, in view of (12), it follows that

$$\sum_{k=1}^{\infty} \frac{[k+|b(A-B)-Bk|]}{|b|(A-B)} \left(\frac{p+k}{p}\right)^\lambda \left\{ \frac{|b|(A-B)}{[k+|b(A-B)-Bk|](\dfrac{p+k}{p})^\lambda} \delta_{p+k} \right\}$$
$$= \sum_{k=1}^{\infty} \delta_{p+k} = 1 - \delta_k \leq 1 \tag{14}$$

So by theorem 2.1, the function $f(z)$ belongs to the class $S_p(A, B, b, \lambda)$. Conversely, let the function $f(z)$ defined by (7) belongs to the class $S_p(A, B, b, \lambda)$. Then

$$a_{p+k} \leq \frac{|b|(A-B)}{[k+|b(A-B)-Bk|](\frac{p+k}{p})^{\lambda}}, p, k \in N \tag{15}$$

Setting

$$\delta_{p+k} = \frac{[k+|b(A-B)-Bk|](\frac{p+k}{p})}{|b|(A-B)} a_{p+k}, p, k \in N \tag{16}$$

and

$$\delta_p = 1 - \sum_{k=1}^{\infty} \delta_{p+k}. \tag{17}$$

We see that $f(z)$ can be expressed in the form (11). This completes the proof of the Theorem 3.1.

Corollary 3.2. *The extreme points of the class $S_p(A, B, b, \lambda)$ are the functions $f_p(z)$ and $f_{p+k}(z)$ given by* (9) *and* (10), *respectively.*

3.2 Modified Hadamard Products

Let the functions $f_i(z) \quad (i = 1, 2)$ be defined by

$$f_i(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,i}, z^{p+k}, p \in N \tag{18}$$

The modified Hadamard product of $f_1(z)$ *and* $f_2(z)$ is defined by

$$\left(f_1 * f_2\right)(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,1} a_{p+k,2} z^{p+k}. \tag{19}$$

Theorem 3.3. *Let $f_i(z) \quad (i = 1, 2)$ defined by* (18) *be in the class $S_p(A, B, b, \lambda)$. Then $(f_1 * f_2) \in S_p(A, B, \alpha, \lambda)$, where*

$$\alpha = p - \frac{(A-B)^2 p^{2\lambda} |b(1+b)|}{[1+|b(A-B)-B|]^2 (p+1)^{\lambda} - (A-B)^2 p^{2\lambda} |b^2|}. \tag{20}$$

The result is sharp.

Proof. To prove the theorem, we need to find the largest α such that

$$\sum_{k=1}^{\infty} \frac{[k+|b(A-\alpha)-\alpha k|](p+k)^{\alpha} |a_{p+k,1} a_{p+k,2}|}{|b|(A-B)p^{\lambda}} \leq 1 \tag{21}$$

Simce
$$\sum_{k=1}^{\infty}\frac{k+|b(A-B)-Bk|(p+k)^{\lambda}}{|b|(A-B)p^{\lambda}}a_{p+k,1}\leq 1. \tag{22}$$
and
$$\sum_{k=1}^{\infty}\frac{k+|b(A-B)-Bk|(p+k)^{\lambda}}{|b|(A-B)p^{\lambda}}a_{p+k,2}\leq 1. \tag{23}$$
By Cauchy-Schwarz inequality, we have
$$\sum_{k=1}^{\infty}\frac{[k+|b(A-B)-Bk|](p+k)^{\lambda}\sqrt{|a_{p+k,1}a_{p+k,2}|}}{|b|(A-B)p^{\lambda}}\leq 1 \tag{24}$$
Thus, it is sufficient to show that
$$\begin{aligned}&\frac{[k+|b(A-\alpha)-\alpha k|](p+k)^{\lambda}}{(A-\alpha)}|a_{p+k,1}a_{p+k,2}|\\ &\leq\frac{[k+|b(A-B)-Bk|](p+k)^{\lambda}}{(A-B)}\sqrt{|a_{p+k,1}a_{p+k,2}|}\end{aligned} \tag{25}$$
i.e. that
$$\sqrt{|a_{p+k,1}a_{p+k,2}|}\leq\frac{[k+|b(A-B)-Bk|](A-\alpha)}{[k+|b(A-\alpha)-\alpha k|](A-B)} \tag{26}$$
Note that
$$\sqrt{|a_{p+k,1}a_{p+k,2}|}\leq\frac{|b|(A-B)p^{\lambda}}{[k+|b(A-B)-Bk|](p+k)^{\lambda}},\, k\in N \tag{27}$$
Consequently, we need only to prove that
$$\frac{|b|(A-B)p^{\lambda}}{[k+|b(A-B)-Bk|](p+k)^{\lambda}}\leq\frac{[k+|b(A-B)-Bk|](A-\alpha)}{[k+|b(A-\alpha)-\alpha k|](A-B)} \tag{28}$$
or equivalently that
$$\alpha\leq p-\frac{|b(1+b)|k(A-B)^{2}p^{2\lambda}}{[k+|b(A-B)-Bk|]^{2}(p+k)^{2\lambda}-|b|(A-B)^{2}p^{2\lambda}} \tag{29}$$
Since
$$A(k)=p-\frac{|b(1+b)|k(A-B)^{2}p^{2\lambda}}{[k+|b(A-B)-Bk|]^{2}(p+k)^{2\lambda}-|b|(A-B)^{2}p^{2\lambda}} \tag{30}$$
is an increasing function of $k(k\geq 1)$ for $\lambda\in N_0$, $-1\leq B<A\leq 1, p\in N$ and b is a non-zero complex number.
Letting $k=1$ in (30), we obtain

$$\alpha \leq p - \frac{|b(1+b)|(A-B)^2 p^{2\lambda}}{[1+|b(A-B)-B|]^2(p+1)^{2\lambda} - |b|(A-B)^2 p^{2\lambda}} \tag{31}$$

which completes the proof of the Theorem 3.3.
Finally, by taking the functions

$$f_i(z) = z^p - \frac{|b|(A-\alpha)p^\lambda}{[1+|b(A-\alpha)-\alpha|](p+1)^\lambda}, i = 1, 2, ...; p \in N. \tag{32}$$

we can see that the result is sharp.

Corollary 3.4. *For $f_i(z)(i = 1, 2)$ as in Theorem 3.3, we have*

$$h(z) = z^p - \sum_{k=1}^{\infty} \sqrt{|a_{p+k,1}a_{p+k,2}|}z^{p+k} \tag{33}$$

belongs to the class $S_p(A, B, b, \lambda)$. The is sharp with the function given by (32).

Proof. The result follows from the inequality (24).

3.3 Inclusion Properties

Theorem 3.5. *Let the functions $f_i(z)$ $(i = 1, 2)$ defined by (18) be in the class $S_p(A, B, b, \lambda)$. Then the function*

$$q(z) = z^p - \sum_{k=1}^{\infty} \left(|a_{p+k,1}a_{p+k,2}|\right) z^{p+k}. \tag{34}$$

belongs to the class $S_p(A, B, \alpha, \lambda)$, where

$$\alpha = p - \frac{2|b(1+b)|(A-B)^2 p^{2\lambda}}{[1+|b(A-B)-B|](p+1)^{2\lambda} - 2|b^2|(A-B)^2 p^{2\lambda}} \tag{35}$$

The result is sharp for the functions $f_i(z)(i = 1, 2)$ defined by (32)

Proof. By the virtue of Theorem 2.2, we obtain

$$\begin{aligned} &\sum_{k=1}^{\infty} \left\{ \frac{[k+|b(A-B)-Bk|](p+k)^\lambda}{|b|(A-B)p^\lambda} \right\}^2 |a^2_{p+k,1}| \\ &\leq \left\{ \sum_{k=1}^{\infty} \frac{[k+|b(A-B)-Bk|](p+k)^\lambda}{|b|(A-B)p^\lambda} |a_{p+k,1}| \right\}^2 \leq 1 \end{aligned} \tag{36}$$

and

$$\sum_{k=1}^{\infty}\left\{\frac{[k+|b(A-B)-Bk|](p+k)^{\lambda}}{|b|(A-B)p^{\lambda}}\right\}^{2}|a_{p+k,2}^{2}| \leq \left\{\sum_{k=1}^{\infty}\frac{[k+|b(A-B)-Bk|](p+k)^{\lambda}}{|b|(A-B)p^{\lambda}}|a_{p+k,2}|\right\}^{2} \leq 1 \tag{37}$$

Therefore, we need to find the largest α such that

$$\frac{[k+|b(A-\alpha)-\alpha k|](p+k)^{\lambda}}{|b|(A-\alpha)p^{\lambda}} \leq \frac{1}{2}\left\{\frac{[k+|b(A-B)-Bk|](p+k)^{\lambda}}{|b|(A-B)p^{\lambda}}\right\}^{2}, k \geq 1 \tag{38}$$

i.e. that

$$\alpha \leq p - \frac{2|b(1+b)|k(A-B)^{2}}{[k+|b(A-B)-Bk|](p+k)^{2\lambda}-2|b^{2}|(A-B)^{2}}, k \geq 1 \tag{39}$$

Since

$$B(k) = p - \frac{2|b(1+b)|k(A-B)^{2}}{[k+|b(A-B)-Bk|](p+k)^{2\lambda}-2|b^{2}|(A-B)^{2}} \tag{40}$$

$$(-1 \leq B < A \leq 1, p \in N);$$

and b is a non-zero complex number, is an increasing function of $k(k \geq 1)$ for $p, \lambda \in N$.
Letting $k = 1$ in (40) we have

$$\alpha = p - \frac{2|b(1+b)|(A-B)^{2}}{[1+|b(A-B)-B|](p+1)^{2\lambda}-2|b^{2}|(A-B)^{2}} \tag{41}$$

which completes the proof of the Theorem 3.5.

Theorem 3.6. *Let the functions $f_i(z)(i = 1, 2)$ defined by (18) be in the class $S_p(A, B, b, \lambda)$. Then the function*

$$\Phi(z) = z^{p} - \frac{1}{m}\sum_{k=1}^{\infty}\sum_{i=1}^{m}|a_{p+k,i}|z^{p+k} \tag{42}$$

belongs to the class $S_p(A, B, \alpha, \lambda)$.

Proof. Since $f_i(z) \in S_p(A, B, b, \lambda)$, by Lemma 2.1 we have

$$\sum_{k=1}^{\infty}\frac{[k+|b(A-B)-Bk|](\frac{p+k}{p})^{\lambda}}{|b|(A-B)}|a_{p+k,i}| \leq 1, i = 1, 2, ..., m \tag{43}$$

So that

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{[k+|b(A-B)-Bk|](p+k)^{\lambda}}{|b|(A-B)p^{\lambda}}\left(\frac{1}{m}\sum_{k=1}^{\infty}|a_{p+k,i}|\right)\\&\leq\frac{1}{m}\sum_{k=1}^{m}\left\{\frac{[k+|b(A-B)-Bk|](p+k)^{\lambda}}{|b|(A-B)p^{\lambda}}\right\}a_{p+k,i}\leq 1\end{aligned}\tag{44}$$

which shows that $f(z)\in S_p(A,B,\alpha,\lambda)$.

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