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On Generalized $(\alpha, \alpha)-$Derivations of $\sigma-$Prime Rings

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Abstract

Let R be a $\sigma-$prime ring. An additive mapping $F : R \to R$ is called a generalized $(\alpha, \alpha)-$ derivation, if there exists a mapping $g : R \to R$ such that $F(xy) = F(x)\alpha(y) + \alpha(x)g(y)$ for all $x, y \in R$. In this paper, some results about generalized derivation are extended to generalized $(\alpha, \alpha)-$derivation in $\sigma-$prime ring.

Keywords: *$\sigma-$prime ring, generalized $(\alpha, \alpha)-$derivation.*

1 Introduction

R will denote an associative ring with center $Z(R)$. Recall that a ring R is *prime ring* if $xRy = (0)$ implies $x = 0$ or $y = 0$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator (bracket) $xy-yx$. An additive mapping $\sigma : R \to R$ is called an *involution* if $(xy)^\sigma = y^\sigma x^\sigma$ and $(x^\sigma)^\sigma = x$ for all $x, y \in R$. Symbol $S_\sigma(R)$ will denote the set of symmetric and skew symmetric elements of R, that is, $S_\sigma(R) = \{x \in R \mid x^\sigma = \pm x\}$. A ring equipped with an involution is called *ring with involution.* A ring with involution is said to *$\sigma-$prime* if $xRy = xRy^\sigma = (0)$ implies that $x = 0$ or $y = 0$. Every prime ring with an involution is a $\sigma-$prime ring but the converse is generally not true. Such an example due to L.Oukhtite [1] is as following. Let R be a prime ring $S = R \times R^o$ where R^o is the opposite ring of R, define $(x, y)^\sigma = (y, x)$ then S is a $\sigma-$prime but not prime. This example shows that every prime ring can be injected in a $\sigma-$prime ring. So, $\sigma-$prime rings are more general than prime rings.

The derivations in prime rings were initiated by E. C. Posner in [2]. Recently, in [3], Bresar defined an additive mapping $F : R \to R$ is called *generalized derivation* if there exists a derivation d such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $F : R \to R$ is called *generalized* $(\alpha, \alpha)-$*derivation* if there exists a mapping $g : R \to R$ such that $F(xy) = F(x)\alpha(y) + \alpha(x) g(y)$ for all $x, y \in R$. So, generalized $(\alpha, \alpha)-$derivation is more general than generalized derivation.

In this paper, some results about generalized derivation of $\sigma-$prime ring are extended to generalized $(\alpha, \alpha)-$derivation in $\sigma-$prime rings.

Throughout this paper, R is a $\sigma-$prime ring, $\alpha : R \to R$ is a surjection mapping, $g : R \to R$ is a mapping and $F : R \to R$ is a nonzero generalized $(\alpha, \alpha)-$derivation with mapping g.

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2 Results

Lemma 2.1. *Let F, g commute with σ and $b \in R$. If $bF(x) = 0$ for all $x \in R$ either $b = 0$ or α is a ring homomorphism of R.*

Proof: From the hypothesis, we get $bF(xy) = 0$ for all $x, y \in R$. By using hypothesis, we get $0 = bF(xy) = b(F(x)\alpha(y) + \alpha(x)g(y)) = bF(x)\alpha(y) + b\alpha(x)g(y) = b\alpha(x)g(y)$. Hence, $b\alpha(x)g(y) = 0$ for all $x, y \in R$. Since α is an onto mapping, one has $bRg(y) = (0)$ for all $y \in R$. Taking y^σ in place of y and using that $g\sigma = \sigma g$, we get $bRg(y)^\sigma = (0)$. Thus, $bRg(y) = (0) = bRg(y)^\sigma$ for all $y \in R$. So, we have

$$b = 0 \text{ or } g(y) = 0, \forall y \in R$$

If $b \neq 0$, we get $F(xy) = F(x)\alpha(y)$ for all $x, y \in R$. Then, we have $F((xy)z) = F(xy)\alpha(z) = F(x)\alpha(y)\alpha(z)$ and $F(x(yz)) = F(x)\alpha(yz)$. If we subtract these two equations from each other, we get

$$\begin{aligned} 0 &= F(x)\alpha(y)\alpha(z) - F(x)\alpha(yz) \\ &= F(x)(\alpha(y)\alpha(z) - \alpha(yz)). \end{aligned}$$

Hence, $F(x)(\alpha(y)\alpha(z) - \alpha(yz)) = 0$ for all $x, y, z \in R$. For an arbitrary $t \in R$, taking xt in place of x, we have $F(x)\alpha(t)(\alpha(y)\alpha(z) - \alpha(yz)) = 0$ for all $x, y, z \in R$. So, we get

$$F(x)R((\alpha(y)\alpha(z) - \alpha(yz)) = (0), \forall x, y, z \in R$$

Replacing x by x^σ in the last equation and noting that $F\sigma = \sigma F$, we have

$$F(x)^\sigma R\left(\alpha(y)\alpha(z) - \alpha(yz)\right) = (0), \forall x, y, z \in R$$

Since F is nonzero, it follows that

$$\alpha(y)\alpha(z) = \alpha(yz) \quad \forall x, y, z \in R$$

On the other hand, we have $F\left(x(y+z)\right) = F(x)\alpha(y+z)$ and $F(xy+xz) = F(xy) + F(xz) = F(x)\alpha(y) + F(x)\alpha(z) = F(x)(\alpha(y) + \alpha(z))$. If we subtract these two equations from each other, we arrive $0 = F(x)\alpha(y+z) - F(x)\left(\alpha(y) + \alpha(z)\right) = F(x)\left(\alpha(y+z) - \alpha(y) - \alpha(z)\right)$ for all $x, y, z \in R$. So, $F(x)\left(\alpha(y+z) - \alpha(y) - \alpha(z)\right) = 0$ for all $x, y, z \in R$. For an arbitrary $k \in R$, taking x in place of xk, it gives $F(x)\alpha(k)\left(\alpha(y+z) - \alpha(y) - \alpha(z)\right) = 0$. Hence,

$$F(x)R\left(\alpha(y+z) - \alpha(y) - \alpha(z)\right) = (0), \forall x, y, z \in R$$

Replacing x by x^σ in the last equation and using F commutes with σ, we get

$$F(x)^\sigma R\left(\alpha(y+z) - \alpha(y) - \alpha(z)\right) = (0), \forall x, y, z \in R$$

Since F is nonzero, it follows $\alpha(y+z) = \alpha(y) + \alpha(z)$ for all $y, z \in R$. So, α is a ring homomorphism of R.

Theorem 2.2. *If α is a ring homomorphism, then g is a $(\alpha, \alpha)-$deriva-tion of R.*

Proof: Based on the definition of F $(\alpha, \alpha)-$derivation, we get

$$\begin{aligned} F\left(z(x+y)\right) &= F(z)\alpha(x+y) + \alpha(z)g(x+y) \\ &= F(z)\alpha(x) + F(z)\alpha(y) + \alpha(z)g(x+y) \end{aligned}$$

and

$$\begin{aligned} F\left(z(x+y)\right) &= F(zx+zy) = F(zx) + F(zy) \\ &= F(z)\alpha(x) + \alpha(z)g(x) + F(z)\alpha(y) + \alpha(z)g(y). \end{aligned}$$

If we subtract these two equations from each other, then

$$\alpha(z)g(x+y) - \alpha(z)g(x) - \alpha(z)g(y) = 0$$

Hence, one has

$$\alpha(z)\left(g(x+y) - g(x) - g(y)\right) = 0, \forall x, y, z \in R$$

Since α is an onto mapping, it gives $R\left(g(x+y)-g(x)-g(y)\right)=(0)$ for all $x,y\in R$. It can hold

$$g(x+y)=g(x)+g(y),\forall x,y\in R$$

On the other hand

$$\begin{aligned}F\left(x(yz)\right)&=F(x)\alpha(yz)+\alpha(x)g(yz)\\&=F(x)\alpha(y)\alpha(z)+\alpha(x)g(yz)\end{aligned}$$

and

$$\begin{aligned}F((xy)z)&=F(xy)\alpha(z)+\alpha(xy)g(z)\\&=F(x)\alpha(y)\alpha(z)+\alpha(x)g(y)\alpha(z)+\alpha(x)\alpha(y)g(z).\end{aligned}$$

If we subtract these two equations from each other, it follows

$$\begin{aligned}0&=\alpha(x)g(yz)-\alpha(x)g(y)\alpha(z)-\alpha(x)\alpha(y)g(z)\\&=\alpha(x)\left(g(yz)-g(y)\alpha(z)-\alpha(y)g(z)\right)\end{aligned}$$

So, $\alpha(x)\left(g(yz)-g(y)\alpha(z)-\alpha(y)g(z)\right)=0$ for all $x,y,z\in R$. This implies that $R\left(g(yz)-g(y)\alpha(z)-\alpha(y)g(z)\right)=(0)$ for all $y,z\in R$. Thus,

$$g(yz)=g(y)\alpha(z)+\alpha(y)g(z),\forall y,z\in R$$

Therefore, g is a $(\alpha,\alpha)-$derivation.

Theorem 2.3. *Let F,g commute with σ and α be onto ring homomorphism. If $F(R)\subset Z$, then R is commutative.*

Proof: From the hypothesis, we get $[F(xy),r]=0$ for all $x,y,r\in R$. If we combine this equation by using the hypothesis, we have

$$F(x)\left[\alpha(y),r\right]+\left[\alpha(x)g(y),r\right]=0,\forall x,y,r\in R \tag{1}$$

Using an arbitrary $z\in R$ and taking $\alpha(y)z$ in place of r, by using the hypothesis, we get

$$\alpha(y)(F(x)\left[\alpha(y),z\right]+\left[\alpha(x)g(y),z\right])+\left[\alpha(x)g(y),\alpha(y)\right]z=0,\forall x,y,z\in R$$

By (1), we obtain $\left[\alpha(x)g(y),\alpha(y)\right]z=0$ for all $x,y,z\in R$. Hence,

$$\left[\alpha(x)g(y),\alpha(y)\right]R=(0)\text{ for all }x,y\in R.$$

It gives $\left[\alpha(x)g(y),\alpha(y)\right]=0$ for all $x,y\in R$. Therefore we get

$$\alpha(x)\left[g(y),\alpha(y)\right]+\left[\alpha(x),\alpha(y)\right]g(y)=0,\forall x,y\in R \tag{2}$$

By (2) equation, using an arbitrary $z \in R$, replacing x by xz and employing (2) equation, one has

$$\begin{aligned} 0 &= \alpha(xz)\left[g(y),\alpha(y)\right] + \left[\alpha(xz),\alpha(y)\right]g(y) \\ &= \alpha(x)(\alpha(z)\left[g(y),\alpha(y)\right] + \left[\alpha(z),\alpha(y)\right]g(y)) + \left[\alpha(x),\alpha(y)\right]\alpha(z)g(y) \\ &= \left[\alpha(x),\alpha(y)\right]\alpha(z)g(y) \end{aligned}$$

Therefore, we obtain $[\alpha(x),\alpha(y)]\,\alpha(z)g(y) = 0$ for all $x, y, z \in R$. It implies that $[\alpha(x),\alpha(y)]\,Rg(y) = (0)$ for all $x, y \in R$. It follows

$$[r,\alpha(y)]\,Rg(y) = (0), \forall r, y \in R \tag{3}$$

For $y \in S_\sigma(R)$ in (3), replacing y by y^σ and by using $g\sigma = \sigma g$, we get $[r,\alpha(y^\sigma)]\,Rg(y^\sigma) = [r,\alpha(y)]\,Rg(y)^\sigma = (0)$ for all $r \in R$, $y \in S_\sigma(R)$. It gives

$$[r,\alpha(y)] = 0 \text{ or } g(y) = 0, \forall r \in R,\ y \in S_\sigma(R) \tag{4}$$

For an arbitrary $t \in R$, $t - t^\sigma \in S_\sigma(R)$. In (4), replacing y by $t - t^\sigma$, we get $[r,\alpha(t - t^\sigma)] = 0$ or $g(t - t^\sigma) = 0$ for all $r, t \in R$. Hence,

$$[r,\alpha(t)] = [r,\alpha(t^\sigma)] \text{ or } g(t) = g(t)^\sigma, \forall r, t \in R$$

Let us define the sets $A = \{t \in R \mid [r,\alpha(t)] = [r,\alpha(t^\sigma)] \text{ for all } r \in R\}$ and $B = \{t \in R \mid g(t) = g(t)^\sigma\}$. Given the fact that a group cannot be the union of two proper subgroups, Brauer's Trick, then $R = A$ or $R = B$. Let us assume that $R = A$. Then, we get

$$[r,\alpha(t)] = [r,\alpha(t^\sigma)]\,, \forall t, r \in R \tag{5}$$

Replacing y by t^σ in (3), we get $[r,\alpha(t^\sigma)]\,Rg(t^\sigma) = [r,\alpha(t^\sigma)]\,Rg(t)^\sigma = (0)$ for all $t, r \in R$. Substituting these values in (5), we obtain $[r,\alpha(t)]\,Rg(t)^\sigma = (0)$ for all $t, r \in R$. Hence, we have

$$[r,\alpha(t)] = 0 \text{ or } g(t) = 0,\ \forall t, r \in R$$

Now, assume that $R = B$. Then $g(t) = g(t^\sigma)$ for all $t \in R$. Replacing y by t in (3), we get $[r,\alpha(t)]\,Rg(t) = (0)$ for all $t, r \in R$. Given that $g(t) = g(t^\sigma)$, we get $[r,\alpha(t)]\,Rg(t)^\sigma = (0)$ for all $t, r \in R$. This means

$$[r,\alpha(t)] = 0 \text{ or } g(t) = 0, \forall t, r \in R$$

In either case, we have the same result. Then, $[r,\alpha(t)]\,Rg(t) = (0)$ for all $t, r \in R$. It follows

$$[r,\alpha(t)] = 0 \text{ or } g(t) = 0,\ \forall t, r \in R.$$

Let us define K and L as follows $K = \{t \in R \mid [r, \alpha(t)] = 0 \text{ for all } r \in R\}$ and $L = \{t \in R \mid g(t) = 0\}$. By Brauer's Trick, $R = K$ or $R = L$. Assume that $R = K$. Then we have $\alpha(R) \subset Z$. So, R is commutative. Assume that $R = L$. Then $g = 0$. So, $F(xy) = F(x)\alpha(y)$ for all $x, y \in R$. By (1) equation, given that $g = 0$ in this equation,

$$F(x)\,[\alpha(y), r] = 0, \forall x, y, r \in R$$

For an arbitrary $t \in R$, replacing x by xt, we get

$$F(xt)\,[\alpha(y), r] = F(x)\alpha(t)\,[\alpha(y), r] = 0.$$

It implies $F(x)R\,[\alpha(y), r] = (0)$. Replacing x by x^σ and noting that $F\sigma = \sigma F$, we obtain

$$F(x)^\sigma R\,[\alpha(y), r] = (0) \text{ for all } x, y, r \in R.$$

As $F(x)R\,[\alpha(y), r] = F(x)^\sigma R\,[\alpha(y), r] = (0)$, one can find that $F(x) = 0$ or $[\alpha(y), r] = 0$ for all $x, y, r \in R$. Since F is nonzero, then R is commutative.

Theorem 2.4. *Let F commute with σ and α be onto ring homomorphism such that commute with σ. If $F([x, y]) = 0$ for all $x, y \in R$, then R is commutative.*

Proof: For $x, y \in R$, replacing x by xy in the hypothesis, we get $F([xy, y]) = F([x, y]\,y) = F([x, y])\alpha(y) + \alpha([x, y])g(y) = \alpha([x, y])g(y) = 0$. Hence, we arrive

$$\alpha([x, y])g(y) = 0, \forall x, y \in R \tag{6}$$

For an arbitrary $t \in R$, replacing x by xt and noting that mapping α is a ring homomorphism, one has $\alpha([xt, y])g(y) = \alpha(x)\alpha([t, y])g(y) + \alpha([x, y])\alpha(t)g(y) = 0$ for all $x, y, t \in R$. By (6), we get

$$\alpha([x, y])Rg(y) = (0), \forall x, y \in R \tag{7}$$

For $y \in S_\sigma(R)$, replacing y by y^σ, we get $\alpha([x, y^\sigma])Rg(y^\sigma) = (0)$. Replacing x by x^σ into the equation, it follows that $\alpha([x^\sigma, y^\sigma])Rg(y^\sigma) = (0)$. As $y \in S_\sigma(R)$, g is an additive mapping and $\alpha\sigma = \sigma\alpha$, we get $\alpha([x, y])^\sigma Rg(y) = (0)$. So, $\alpha([x, y])Rg(y) = \alpha([x, y])^\sigma Rg(y) = (0)$ for all $x \in R$ and $y \in S_\sigma(R)$. Hence,

$$\alpha([x, y]) = 0 \text{ or } g(y) = 0, \forall x \in R, y \in S_\sigma(R) \tag{8}$$

Assume that $t \in R$. Then, $t - t^\sigma \in S_\sigma(R)$. Replacing y by $t - t^\sigma$ in (8) one can find for all $x, t \in R$

$$\alpha([x, t]) = \alpha([x, t^\sigma]) \text{ or } g(t) = g(t^\sigma)$$

Let us define the sets $A = \{t \in R \mid \alpha([x,t]) = \alpha([x,t^\sigma]),\text{ for all } x \in R\}$ and $B = \{t \in R \mid g(t) = g(t^\sigma)\}$. From Brauer's Trick $R = A$ or $R = B$. Assume that $R = A$. Replacing y by t in (7), we get $\alpha([x,t])Rg(t) = (0)$. Employing that $\alpha([x,t]) = \alpha([x,t^\sigma])$, we find $\alpha([x,t^\sigma])Rg(t) = (0)$. Replacing x by x^σ, it gives $\alpha([x^\sigma,t^\sigma])Rg(t) = (0)$. So we get

$$\alpha([x,t])^\sigma Rg(t) = (0)$$

Since $\alpha([x,t])^\sigma Rg(t) = (0) = \alpha([x,t])Rg(t) = (0)$ for all $x,t \in R$, we have $\alpha([x,t]) = 0$ or $g(t) = 0$. Now assume $R = B$. Then, we get $g(t) = g(t^\sigma)$ for all $t \in R$. Replacing y by t in (7), we conclude that $\alpha([x,t])Rg(t) = (0)$. Taking x^σ in place of x and t^σ in place of t, we get $\alpha([x,t])^\sigma Rg(t^\sigma) = (0)$. Employing that $g(t) = g(t^\sigma)$, it gives $\alpha([x,t])^\sigma Rg(t) = (0)$. It follows that $\alpha([x,t]) = 0$ or $g(t) = 0$ for all $x,t \in R$. In either $R = A$ or $R = B$, we have the same result

$$\alpha([x,t]) = 0 \text{ or } g(t) = 0 \text{ for all } x,t \in R.$$

Let us define the sets $K = \{t \in R \mid \alpha([x,t]) = 0,\text{ for all } x \in R\}$ and $L = \{t \in R \mid g(t) = 0\}$. Now applying Brauer's Trick for this sets, we have $R = K$ or $R = L$. Assume that $R = K$. Then we get $0 = \alpha([x,t]) = [\alpha(x),\alpha(t)]$ for all $x,t \in R$. Hence, R is commutative. Assume that $R = L$. That is $g = 0$. This yields $F(xy) = F(x)\alpha(y)$ for all $x,y \in R$. For an arbitrary $z \in R$, replacing x by xz in the hypothesis, we obtain $0 = F([xz,y]) = F(x)\alpha([z,y]) + F([x,y])\alpha(z)$. By the hypothesis, we get $0 = F(x)\alpha([z,y]) + F([x,y])\alpha(z) = F(x)\alpha([z,y])$. This yields, $F(x)\alpha([z,y]) = 0$ for all $z,y,x \in R$. Replacing x by xt for an arbitrary $t \in R$, it gives $0 = F(xt)\alpha([z,y]) = F(x)\alpha(t)\alpha([z,y])$ for all $z,y,x,t \in R$. It means $F(x)R\alpha([z,y]) = (0)$. Replacing x by x^σ, one can find $F(x^\sigma)R\alpha([z,y]) = 0$. So we get $F(x)^\sigma R\alpha([z,y]) = (0)$. As $F(x)R\alpha([z,y]) = (0) = F(x)^\sigma R\alpha([z,y])$ for all $z,y,x \in R$ and F is nonzero, it implies $\alpha([z,y]) = 0$ for all $z,y,x \in R$. Hence R is commutative.

Theorem 2.5. *Let g commute with σ and α be onto ring homomorphism. If $F([x,y]) = [y,\alpha(x)]$ for all $x,y \in R$, then $F(y) = -y$ or R is commutative.*

Proof: For an arbitrary $y,x \in R$. Replacing x by xy in the hypothesis, we get $F([xy,y]) = [y,\alpha(xy)]$.

$$[y,\alpha(xy)] = F([xy,y]) = F([x,y]\,y) = F([x,y])\alpha(y) + \alpha([x,y])g(y)$$

Since α is a ring homomorphism, we arrive $F([x,y])\alpha(y) + \alpha([x,y])g(y) = [y,\alpha(xy)] = [y,\alpha(x)\alpha(y)] = \alpha(x)\,[y,\alpha(y)] + [y,\alpha(x)]\,\alpha(y)$. It gives $F([x,y])\alpha(y) + \alpha([x,y])g(y) = \alpha(x)\,[y,\alpha(y)] + [y,\alpha(x)]\,\alpha(y)$ for all $x,y \in R$. By the hypothesis, one can find $[y,\alpha(x)]\,\alpha(y) + \alpha([x,y])g(y) = \alpha(x)\,[y,\alpha(y)] + [y,\alpha(x)]\,\alpha(y)$. When we simplify the equation, we get

$$\alpha(x)\,[y,\alpha(y)] - \alpha([x,y])g(y) = 0, \forall y,x \in R \tag{9}$$

For an arbitrary $z \in R$, replacing x by xz, we obtain

$$\begin{aligned} 0 &= \alpha(xz)\,[y,\alpha(y)] - \alpha([xz,y])g(y) \\ &= \alpha(x)(\alpha(z)\,[y,\alpha(y)] - \alpha([z,y])g(y)) - \alpha\,([x,y])\alpha(z)\,g(y) \end{aligned}$$

Replacing x by z in (9) and put in the last equation, we get

$$\begin{aligned} 0 &= \alpha(x)(\alpha(z)\,[y,\alpha(y)] - \alpha([z,y])g(y)) - \alpha\,([x,y])\alpha(z)\,g(y) \\ &= \alpha\,([x,y])\alpha(z)\,g(y). \end{aligned}$$

So, we get

$$\alpha([x,y])Rg(y) = (0), \forall x, y \in R \tag{10}$$

For an arbitrary $y \in S_\sigma(R)$, replacing y by y^σ, one can find

$$\alpha([x,y^\sigma])Rg(y^\sigma) = (0).$$

As $y^\sigma = \pm y$ and $g\sigma = \sigma g$, we get $\alpha([x,y])Rg(y)^\sigma = (0)$. It gives

$$\alpha([x,y]) = 0 \text{ or } g(y) = 0, \forall x \in R, y \in S_\sigma(R) \tag{11}$$

For an arbitrary $t \in R$. Then, $t - t^\sigma \in S_\sigma(R)$. Replacing y by $t - t^\sigma$ in (11), we find that for all $x, t \in R$

$$\alpha([x,t]) = \alpha([x,t^\sigma]) \text{ or } g(t) = g(t^\sigma)$$

Let us define the sets $A = \{t \in R \mid \alpha([x,t]) = \alpha([x,t^\sigma]),\ for\ all\ x \in R\}$ and $B = \{t \in R \mid g(t) = g(t^\sigma)\}$ sets. By Brauer's Trick $R = A$ or $R = B$. Assume that $R = A$. Replacing y by t^σ in (10), we get

$$\alpha([x,t^\sigma])Rg(t^\sigma) = (0).$$

Noting that $\alpha([x,t]) = \alpha([x,t^\sigma])$, we get $\alpha([x,t])Rg(t^\sigma) = (0)$. Given that $g\sigma = \sigma g$, $\alpha([x,t])Rg(t)^\sigma = (0)$. It gives $\alpha([x,t]) = 0$ or $g(t) = 0$ for all $x, t \in R$. Assume that $R = B$. Then, $g(t) = g(t)^\sigma$ for all $t \in R$. Replacing y by t in (10), substituting the value into this equation, we get $\alpha([x,t])Rg(t)^\sigma = (0)$. Since $\alpha([x,t])Rg(t)^\sigma = (0) = \alpha([x,t])Rg(t)$ for all $x, t \in R$, it implies that $\alpha([x,t]) = 0$ or $g(t) = 0$. Then, in either case, we have the same result. Hence,

$$\alpha([x,t]) = 0 \text{ or } g(t) = 0 \text{ for all } x, t \in R$$

Let us define the sets $K = \{t \in R \mid \alpha([x,t]) = 0, \text{ for all } x \in R\}$ and $L = \{t \in R \mid g(t) = 0\}$. By Brauer's Trick $R = L$ or $R = K$. Assume that $R = K$. So, we get $\alpha([x,t]) = [\alpha(x), \alpha(t)] = 0$. Since this equation holds for all $x \in R$, one has $[\alpha(R), \alpha(t)] = (0)$. Thus R is commutative. Assume that $R = L$. Then $g = 0$. That is, $F(xy) = F(x)\alpha(y)$ for all $x, y \in R$. Replacing x by xy in the hypothesis, it follows $F([xy, y]) = [y, \alpha(xy)]$. By using $F(xy) = F(x)\alpha(y)$ and $F([x, y]) = [y, \alpha(x)]$, we have $\alpha(x)[y, \alpha(y)] = 0$ for all $x, y \in R$. It means $R[y, \alpha(y)] = (0)$ for all $y \in R$. It follows that

$$[y, \alpha(y)] = 0, \forall y \in R \tag{12}$$

Replacing y by yx in the hypothesis, we get $F([x, yx]) = [yx, \alpha(x)]$ for all $x, y \in R$. So we get $F(y[x, x] + [x, y]x) = F([x, y]x) = y[x, \alpha(x)] + [y, \alpha(x)]x$. It implies $F([x, y])\alpha(x) = y[x, \alpha(x)] + [y, \alpha(x)]x$. By the hypothesis, it follows from that $[y, \alpha(x)]\alpha(x) = y[x, \alpha(x)] + [y, \alpha(x)]x$. As $[x, \alpha(x)]$ in (12), then $[y, \alpha(x)]\alpha(x) = [y, \alpha(x)]x$. Rewriting this equation, we get

$$[y, \alpha(x)](\alpha(x) - x) = 0, \forall x, y \in R \tag{13}$$

For an arbitrary $z \in R$, replacing y by yz in (13), one can find

$$\begin{aligned} 0 &= [yz, \alpha(x)](\alpha(x) - x) \\ &= y[z, \alpha(x)](\alpha(x) - x) + [y, \alpha(x)]z(\alpha(x) - x). \end{aligned}$$

Employing (13) here, one has $[y, \alpha(x)]z(\alpha(x) - x) = 0$. So we get

$$[y, \alpha(x)]R(\alpha(x) - x) = (0) \tag{14}$$

For any $x \in S_\sigma(R)$, replacing x by x^σ in (14), we obtain

$$[y, \alpha(x^\sigma)]R(\alpha(x^\sigma) - x^\sigma) = (0)$$

So we get $[y, \alpha(x)]R(\alpha(x) - x)^\sigma = (0)$ for all $y \in R$ and $x \in S_\sigma(R)$. It gives

$$[y, \alpha(x)] = 0 \text{ or } \alpha(x) - x = 0, \forall y \in R, x \in S_\sigma(R) \tag{15}$$

For an arbitrary $t \in R$, $t - t^\sigma \in S_\sigma(R)$. Replacing x by $t - t^\sigma$ in (15), we obtain

$$[y, \alpha(t)] = [y, \alpha(t^\sigma)] \, or \, (\alpha(t) - t) = (\alpha(t) - t)^\sigma, \forall y, t \in R$$

Now, let us define the sets $D = \{t \in R \mid [y, \alpha(t)] = [y, \alpha(t^\sigma)], \forall y \in R\}$ and $E = \{t \in R \mid (\alpha(t) - t) = (\alpha(t) - t)^\sigma\}$. From Brauer's Trick, $R = D$ or $R = E$. Assume that $R = D$. Replacing x by t in (14), one has $[y, \alpha(t)]R(\alpha(t) - t) = (0)$. Noting that $[y, \alpha(t)] = [y, \alpha(t)^\sigma]$, we get $[y, \alpha(t)^\sigma]R(\alpha(t) - t) = (0)$. Replacing y by y^σ, one can find $[y^\sigma, \alpha(t)^\sigma]R(\alpha(t) - t) = (0)$. It gives

$$[y, \alpha(t)]^\sigma R(\alpha(t) - t) = (0)$$

and so

$$[y, \alpha(t)] = 0 \text{ or } \alpha(t) - t = 0 \quad \forall t, y \in R.$$

Assume that $R = E$. For all $t \in R$, $\alpha(t) - t = (\alpha(t) - t)^{\sigma}$. Replacing x by t in (13), one has $[y, \alpha(t)] R (\alpha(t) - t) = (0)$. If we substitute this value into the last equation we obtain $[y, \alpha(t)] R (\alpha(t) - t)^{\sigma} = (0)$. It holds $[y, \alpha(t)] = 0$ or $\alpha(t) - t = 0$ for all $y, t \in R$. Thus, in either $R = D$ or $R = E$, we decude the same result

$$[y, \alpha(t)] = 0 \text{ or } \alpha(t) - t = 0, \forall y, t \in R.$$

Define $U = \{t \in R \mid [y, \alpha(t)] = 0, \ for \ all \ x \in R\}$ and $V = \{t \in R \mid \alpha(t) = t\}$. By Brauer's Trick $R = U$ or $R = V$. Assume that $R = U$. So we get $[R, \alpha(t)] = (0)$ for all $t \in R$. It means R is commutative. Assume that $R = V$. It follows that $\alpha(t) = t$ for all $t \in R$. That is, α is identity mapping. Therefore $F(xy) = F(x)\alpha(y) = F(x)y$ and $F([x, y]) = [y, \alpha(x)] = [y, x]$ for all $x, y \in R$. For an arbitrary $z \in R$, replacing y by yz in the hypothesis, then $F([x, yz]) = [yz, x]$. By using $F([x, y]) = [y, x]$ we have $F(y)[x, z] = y[z, x]$. One can obtain

$$(F(y) + y)[x, z] = 0, \forall x, y, z \in R \tag{16}$$

Provided that $r \in R$ in (16), replacing x by xr, one can find $(F(y)+y)x[r, z] = 0$. So $(F(y) + y)R[r, z] = (0)$. Replacing r by r^{σ} and z by z^{σ}, we get $(F(y) + y)R[r, z]^{\sigma} = (0)$. So, we have $F(y) + y = 0$ or $[r, z] = 0$. That is, either $F(y) = -y$ for all $y \in R$ or R is commutative. This completes the proof.

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