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Rough Approximations in a Topological Group

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Abstract

In this paper, we shall introduce lower and upper approximation operators of rough sets are applied into topological group theory and so the notion of a topological rough subgroup has been introduced. Then, we present some propositions and continue to study the image and inverse image of rough approximations of a subgroup with respect to a continuous homomorphism between two topological groups.

Keywords: *Topological groups, Rough sets, Lower approximation, Upper approximation, Rough group.*

1 Introduction

The rough set theory was firstly proposed by Pawlak [18] in 1982 as an effective mathematical tool for modeling and processing incomplete information. Important applications of the rough set theory have been applied in many science fields, for example in medical science, data analysis, knowledge discovery in database [19, 20, 22, 27].

The rough set theory deals with the approximation of an arbitrary subset of a universe is described by two definable subsets called lower and upper approximations. The Pawlak rough set approximations are defined by an equivalence relation.The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and the upper approximation is the union of all the equivalence classes which have a non-empty intersection with the set. In recent years, the rough set theory has been combined with some mathematical theories such as topology and algebra [1, 8, 9, 10, 13, 14, 17, 21, 25, 26, 31, 32, 33]. Many advanced algebraic properties of rough sets were presented in the literature, for example, Bonikowaski [3], Iwinski [7], and Pomykala and Pomykala [23] studied algebraic properties of rough sets. Biswas and Nanda [2] introduced the notion of rough group and rough subgroups that their notion depends on the upper approximation and does not depend on the lower approximation. In [16], Miao et al. improve definitions of rough group and rough subgroup, and prove their new properties. Kuroki and Wang [11] presented some properties of the lower and upper approximations with respect to the normal subgroups. In addition, some properties of the lower and the upper approximations with respect to the normal subgroups were studied in [4, 15, 28, 29, 30]. Also, Kuroki in [12], introduced the notion of a rough ideal in a semigroup. Davvaz in [5], introduced the notion of rough subring with respect to an ideal of a ring. In [6], Davvaz and Mahdavipour considered rough modules.

In this article we generalize to topological rough subgroups the concept and relations which have been established for rough subgroups in [11] and [28]. In section 2, we review some basic notions of rough groups and topological groups. In section 3, we show that lower and upper approximations in a topological group and some properties of them are proved. In section 4, we further study the properties of topological rough subgroups in a topological group. In section 5, we study homomorphic images of topological rough approximations of a topological subgroup. Finally, our conclusions are presented.

2 Preliminaries

In this section, we give some basic definitions and results which will be used later on [11, 24]. Throughout this paper, G denotes a finite group with identity e unless stated otherwise.

Notation 2.1 *If X and Y are two subsets of a group G, we denote by XY the subset composed of all the elements of the form xy, where $x \in X$, $y \in Y$. We denote by X^{-1} the subset composed of all the elements of the form x^{-1}, where $x \in X$.*

Definition 2.2 *Let G be a group and R be an equivalence relation on G. Then R is called a congruence relation of G if R satisfies the following condition:*

$$\forall x \in G, (a, b) \in R \Rightarrow (ax, bx), (xa, xb) \in R.$$

Definition 2.3 *Let N be a normal subgroup of G and X a nonempty subset of G. Let*

$$\underline{N}(X) := \{x \in G : xN \subseteq X\} = \bigcup_{x \in G} \{xN : xN \subseteq X\}$$

,

$$\overline{N}(X) := \{x \in G : xN \cap X \neq \varnothing\} = \bigcup_{x \in G} \{xN : xN \cap X \neq \varnothing\}.$$

Then $\underline{N}(X)$ and $\overline{N}(X)$ are called lower and upper approximations of X with respect to the normal subgroup N, respectively.

Definition 2.4 *Let N be a normal subgroup of a group G and X a nonempty subset of G. Then X is called a upper rough subgroup (respectively, normal subgroup) of G if $\overline{N}(X)$ is a subgroup (respectively, normal subgroup) of G. Similarly, X is called a lower rough subgroup (respectively, normal subgroup) of G if $\underline{N}(X)$ is a subgroup (respectively, normal subgroup) of G.*

Definition 2.5 *A set G of elements is called a topological group if*

(1) *G is an abstract group,*

(2) *G is a topological space,*

(3) *the group operations in G are continuous in the topological space G.*

In greater detail this condition can be formulated as follows:

($tg1$) If x and y are two elements of the group G, then for every neighborhood W of the element xy there exist neighborhoods U and V of the elements x and y such that $UV \subseteq W$.

($tg2$) If x is an element of the group G, then for every neighborhood W of the element x^{-1} there exist neighborhood V of the element x such that $V^{-1} \subseteq W$.

or

($tg3$) If x and y are two elements of the group G, then for every neighborhood W of the element xy^{-1} there exist neighborhoods U and V of the elements x and y such that $UV^{-1} \subseteq W$.

Let G be a topological group and let H be a subgroup of G. Then H becomes a topological group when endowed with the topology induced by G.

Let G_1 and G_2 be two topological groups and $f : G_1 \to G_2$ is a onto homomorphism. If f be a continuous homomorphism, then f is called a topological homomorphism. If f is simultaneously an isomorphism and a homeomorphism, then f is called a topological isomorphism. If $f : G_1 \to f(G_1) \subseteq G_2$ is a topological homomorphism, where $f(G_1)$ carries the topology induced by G_2, then f is called topological group embedding. Suppose N and H are subgroups of G_1 and G_2, respectively. We know that $f(N)$ and $f^{-1}(H)$ are subgroups of G_2 and G_1, respectively. If N and H are normal, then $f(N)$ and $f^{-1}(H)$ are also normal.

Definition 2.6 *Let N be a subgroup of a topological group G. The quetient topology on G/N is defined such that a set $U \subset G/N$ is open if and only if $\rho^{-1}(U)$ is open in the topology of G, where $\rho : G \to G/N$ is the canonical projection map.*

Theorem 2.7 *Let N be a normal subgroup of a topological group G. Let G/N be equipped with the quotient topology. Then*

(1) *The canonical projection $\rho : G \to G/N$ is a open homomorhipsm,*

(2) *G/N is a topological group.*

Remark 2.8 *Let G be a topological group and N be a normal subgroup of G . Let $\rho : G \to G/N$ be the canonical homomorphism. Then*

$$\text{If } U \subset G, \text{ then } \rho^{-1}(\rho(U)) = UN.$$

Theorem 2.9 *Let G_1 and G_2 be two groups. Let $f : G_1 \to G_2$ be an open onto topological homomorphism, with $Kerf = N$. Then, the the map $\varphi : G_1/N \to G_2$, is a topological isomorphism.*

Theorem 2.10 *Let G_1 and G_2 be two groups. Let $f : G_1 \to G_2$ be an onto topological homomorphism, with $N_2 \lhd G_2$ and $N_1 = f^{-1}(N_2)$. Then*
If f is an open, then G_1/N_1 is topological isomorphic to G_2/N_2.

3 Lower and Upper Approximations in a Topological Group

In this section we introduce the concept of lower and upper approximations in a topological group. Then, we give some properties of them.

Definition 3.1 *Let G be a topological group. Let N be a normal subgroup of G and X a nonempty subset of G. Then the sets*

$$\underline{N}(X) := \{x \in G : xN \subseteq X\} = \bigcup_{x\in G} \{xN : xN \subseteq X\}$$

,

$$\overline{N}(X) := \{x \in G : xN \cap X \neq \varnothing\} = \bigcup_{x\in G} \{xN : xN \cap X \neq \varnothing\}.$$

are called lower and upper approximations of the set X with respect to the normal subgroup N, respectively.

Proposition 3.2 *Let N and H be normal subgroups of a topological group G. Let X and Y be any nonempty subsets of G. Then*

(1) $\underline{N}(X) \subseteq X \subseteq \overline{N}(X)$,

(2) $\underline{N}(\varnothing) = \overline{N}(\varnothing) = \varnothing$, $\underline{N}(G) = \overline{N}(G) = G$,

(3) $\overline{N}(X \cup Y) = \overline{N}(X) \cup \overline{N}(Y)$,

(4) $\underline{N}(X \cap Y) = \underline{N}(X) \cap \underline{N}(Y)$,

(5) $\underline{N}(X \cup Y) \supseteq \underline{N}(X) \cup \underline{N}(Y)$,

(6) $\overline{N}(X \cap Y) \subseteq \overline{N}(X) \cap \overline{N}(Y)$,

(7) $X \subset Y \Rightarrow \underline{N}(X) \subseteq \underline{N}(Y)$ *ve* $\overline{N}(X) \subseteq \overline{N}(Y)$,

(8) $\underline{N}(-X) \subseteq -\overline{N}(X)$,

(9) $-\underline{N}(X) \subseteq \overline{N}(-X)$,

(10) $\underline{N}\underline{N}(X) = \overline{N}\underline{N}(X) = \underline{N}(X)$,

(11) $\overline{N}\overline{N}(X) = \underline{N}\overline{N}(X) = \overline{N}(X)$

Proof: The proof is similar to Proposition 2.1 of [11].

Proposition 3.3 *Let N be a normal subgroups of a topological group G and X be any nonempty subsets of G. Then*

İf N is open (closed), then lower and upper approximations of X are open (closed).

Proof: Let N be open in G . Then xN is open. Thus, by the definition of lower approximations of X we have $\underline{N}(X)$ is open. Similarly, $\overline{N}(X)$ is open. Similarly, if N is closed , then lower and upper approximations of X are closed.

Now, we give an example of the lower and upper approximations theory applied to the topological group theory.

Example 3.4 *Let* $G = Z_4 = \left\{\overline{0}, \overline{1}, \overline{2}, \overline{3}\right\}$ *be a set of surplus class with respect to module 4. Let* $G = Z_4$ *a topological group with topology* $T_G = \left\{\varnothing, G, \left\{\overline{0}, \overline{2}\right\}, \left\{\overline{1}, \overline{3}\right\}\right\}$ *and* $N = \left\{\overline{0}, \overline{2}\right\}$ *be a normal subgroups of* G*. Let* $X = \left\{\overline{0}, \overline{2}, \overline{3}\right\}$*. Then by the definition of rough approximations we have* $\underline{N}(X) = \left\{\overline{0}, \overline{2}\right\}$ *and* $\overline{N}(X) = \left\{\overline{0}, \overline{1}, \overline{2}, \overline{3}\right\}$*. Since* $N = \left\{\overline{0}, \overline{2}\right\}$ *is open then* $\underline{N}(X) = \left\{\overline{0}, \overline{2}\right\}$ *and* $\overline{N}(X) = \left\{\overline{0}, \overline{1}, \overline{2}, \overline{3}\right\}$ *are open.*

In [28] C.Z. Wang and D.G. Chen prove the next two propositions for the lower approximations of a subgroup and the upper approximations of a subset in a group. In the following we rewrite them in topological group.

Proposition 3.5 *Let* N *be a normal subgroups of a topological group* G *and* X *a subgroup of* G*. If* $N \subsetneq X$*, then* $\underline{N}(X) = \varnothing$*; if* $N \subseteq X$*, then* $\underline{N}(X) = X$*.*

Proof: The proof is similar to Theorem 3.2. of [28].

Proposition 3.6 *Let* N *be a normal subgroups of a topological group* G *and* X *be any nonempty subset of* G*. Then*

$$\overline{N}(X) = XN.$$

Proof: The proof is similar to Theorem 3.1. of [28].

Corollary 3.7 *Let* N *and* X *be normal subgroups of a topological group* G *and* $N \subseteq X$*. Then*

$$\underline{N}(X) = \overline{N}(X) = X.$$

Proof: By Proposition 3.5 we have $\underline{N}(X) = X$. Since $N \subseteq X$ and from Proposition 3.6 $\overline{N}(X) = X$. Hence $\underline{N}(X) = \overline{N}(X) = X$.

In [11] N.Kuroki and P.P. Wang prove the next four propositions for the rough approximations in a group. In the following we rewrite them in topological group.

Proposition 3.8 *Let* N *be a normal subgroups of a topological group* G*. Let* X *and* Y *be nonempty subsets of* G*. Then*

(1) $\overline{N}(X)\overline{N}(Y) = \overline{N}(XY)$,

(2) $\underline{N}(X)\underline{N}(Y) \subseteq \underline{N}(XY)$.

Proof: (1) The proof is similar to Proposition 2.2. of [11].
(2) The proof is similar to Proposition 2.3. of [11].

Proposition 3.9 *Let N and H be normal subgroups of a topological group G. Let X be a nonempty subset of G. Then*

(1) $\left(\overline{H\cap N}\right)(X)=\overline{H}(X)\cap\overline{N}(X)$,

(2) $\left(\underline{H\cap N}\right)(X)=\underline{H}(X)\cap\underline{N}(X)$.

Proof: (1) The proof is similar to Proposition 2.4. of [11].
(2) The proof is similar to Proposition 2.5. of [11].

4 Lower and Upper Topological Rough Subgroups in a Topological Group

In this section we introduce the concept of lower and upper topological rough subgroups in a topological group. Then, we give some properties of them.

Definition 4.1 *Let N be a normal subgroup of a topological group G and X a nonempty subset of G.*

Then X is called a upper topological rough subgroup (respectively, normal subgroup) of G if $\overline{N}(X)$ is a topological subgroup (respectively, normal subgroup) of G. Similarly, X is called a lower topological rough subgroup (respectively, normal subgroup) of G if $\underline{N}(X)$ is a topological subgroup (respectively, normal subgroup) of G.

Proposition 4.2 *Let N be a normal subgroup of a topological group G. If X is a subgroup of G, then X is a upper topological rough subgroup of G.*

Proof: By Proposition 3.1. in [11] X is a upper rough subgroup of G.

Now, we show that X is a a upper topological rough subgroup of G. Let x and y are two elements of the group $\overline{N}(X)$ and let $xy^{-1}=z$. Every neighborhood W' of the element z in the space $\overline{N}(X)$ can be obtained as the intersection with $\overline{N}(X)$ of some neighborhood W of the element z in the space G, $W'=W\cap\overline{N}(X)$, where $W'N\cap X\neq\varnothing$. Since G topological group, there exist neighborhoods U and V of the elements x and y such that için $UV^{-1}\subseteq W$. Now $U'=U\cap\overline{N}(X)$ and $V'=V\cap\overline{N}(X)$ are relative neighborhoods of the elements x and y in the space $\overline{N}(X)$, where $U'N\cap X\neq\varnothing$ and $V'N\cap X\neq\varnothing$. Thus we have

$$U'V'^{-1}\subseteq UV^{-1}\subseteq W,$$

and also

$$U'V'^{-1}\subseteq\overline{N}(X)\left(\overline{N}(X)\right)^{-1}=\overline{N}(X)\ \text{and}\ \left(U'V'^{-1}\right)N\cap X\neq\varnothing.$$

Hence

$$U^{'}V^{'-1} \subseteq W \cap \overline{N}(X) = W^{'} \text{ and } W^{'}N \cap X \neq \varnothing,$$

i.e., condition $(tg3)$ of Definition 2.5 is satisfied for the group $\overline{N}(X)$ .Therefore X is a upper topological rough subgroup of G.

Proposition 4.3 *Let N be a normal subgroup of a topological group G. If X is a normal subgroup of G, then X is a upper topological rough normal subgroup of G*

Proof: We already know by Proposition 4.2 that X is a upper topological rough subgroup G. It suffices to show that $\overline{N}(X)$ is normal. Let a and x be any elements of $\overline{N}(X)$ and G, respectively. Then there exists an element y in G such that $y \in aN \cap X$, that is, $y \in aN$ and $y \in X$. Since N is normal,

$$xyx^{-1} \in x(aN)x^{-1} = (xa)(Nx^{-1}) = (xa)(x^{-1}N) = (xax^{-1})N.$$

Since X is normal,

$$xyx^{-1} \in xXx^{-1} \subseteq X.$$

Thus $xyx^{-1} \in (xax^{-1})N \cap X$, and so $xax^{-1} \in \overline{N}(X)$. This means that $\overline{N}(X)$ is normal.

Proposition 4.4 *Let N be a normal subgroup of a topological group G. If X is a subgroup of G and $N \subseteq X$, then X is a lower topological rough subgroup of G.*

Proof: By Proposition 3.3. in [11] X is a lower rough subgroup of G.

Now, we show that X is a lower topological rough subgroup of G. Let x and y are two elements of the group $\underline{N}(X)$ and let $xy^{-1} = z$. Every neighborhood $W^{'}$ of the element z in the space $\underline{N}(X)$ can be obtained as the intersection with $\underline{N}(X)$ of some neighborhood W of the element z in the space G, $W^{'} = W \cap \underline{N}(X)$, where $W^{'}N \subseteq X$. Since G topological group, there exist neighborhoods U and V of the elements x and y such that için $UV^{-1} \subseteq W$. Now $U^{'} = U \cap \underline{N}(X)$ and $V^{'} = V \cap \underline{N}(X)$ are relative neighborhoods of the elements x and y in the space $\underline{N}(X)$, where $U^{'}N \subseteq X$ and $V^{'}N \subseteq X$. Thus we have

$$U^{'}V^{'-1} \subseteq UV^{-1} \subseteq W,$$

and also

$$U^{'}V^{'-1} \subseteq \underline{N}(X)\left(\underline{N}(X)\right)^{-1} = \overline{N}(X) \text{ and } \left(U^{'}V^{'-1}\right)N \subseteq X.$$

Hence

$$U^{'}V^{'-1} \subseteq W \cap \underline{N}(X) = W^{'} \text{ and } W^{'}N \subseteq X,$$

i.e., condition $(tg3)$ of Definition 2.5 is satisfied for the group $\underline{N}(X)$ .Therefore X is a lower topological rough subgroup G.

Proposition 4.5 *Let N be a normal subgroup of a topological group G. If X is a normal subgroup of G and $N \subseteq X$, then X is a lower topological rough normal subgroup of G.*

Proof: We already know by Proposition 4.4 that X is a lower topological rough subgroup G. It suffices to show that $\underline{N}(X)$ is normal. Let a and x be any elements of $\underline{N}(X)$ and G, respectively. Then $aN \subseteq X$. Since N and X are normal,

$$(xax^{-1})N = x(aN)x^{-1} \subseteq xXx^{-1} \subseteq X$$

and so $xax^{-1} \in \underline{N}(X)$, which means that $\underline{N}(X)$ is normal.

Proposition 4.6 *Let N be a normal subgroups of a topological group G and X be a subgroup of G. Then*

(1) If $\underline{N}(X)$ a subgroup of G is open, then $\underline{N}(X)$ is also closed.

(2) If $\overline{N}(X)$ a subgroup of G is open, then $\overline{N}(X)$ is also closed.

Proof: (1) Let $\underline{N}(X)$ be open in G, $a \in (\underline{N}(X))^{-}$ and let us prove that $\underline{N}(X) = (\underline{N}(X))^{-}$. Since $\underline{N}(X)$ is open and $e \in \underline{N}(X)$, then $a(\underline{N}(X))$ is open and $a \in a(\underline{N}(X))$ .Thus $a(\underline{N}(X)) \cap \underline{N}(X) \neq \varnothing$. Then, there exist $b \in G$ such that $b \in a(\underline{N}(X))$ and $b \in \underline{N}(X)$.

If $b \in a(\underline{N}(X))$, then there exist $h \in \underline{N}(X)$ such that $hN \subset X$ and $b = ah$.

Now,

$$\text{if } b \in \underline{N}(X), \text{ then } bN \subset X$$

Thus, since $b, h \in \underline{N}(X)$ we have

$$bh^{-1} \in \underline{N}(X) \Rightarrow \left(bh^{-1}\right) N \subset X \Rightarrow aN \subset X ,$$

and so

$$aN \subset X \Rightarrow a \in \underline{N}(X) \Rightarrow (\underline{N}(X))^{-} \subseteq \underline{N}(X).$$

Hence

$$\underline{N}(X) = (\underline{N}(X))^{-}.$$

This prove $\underline{N}(X)$ is closed.

(2) Let $\overline{N}(X)$ be open in G, $a \in \left(\overline{N}(X)\right)^{-}$ and let us prove that $\overline{N}(X) = \left(\overline{N}(X)\right)^{-}$. Since $\overline{N}(X)$ is open and $e \in \overline{N}(X)$, then $a\left(\overline{N}(X)\right)$ is open and $a \in a\left(\overline{N}(X)\right)$ .Thus $a\left(\overline{N}(X)\right) \cap \overline{N}(X) \neq \varnothing$. Then, there exist $b \in G$ such that $b \in a\left(\overline{N}(X)\right)$ and $b \in \overline{N}(X)$. If

$b \in a\left(\overline{N}(X)\right)$, then there exist $h \in \overline{N}(X)$ such that $hN \cap X \neq \varnothing$ and $b = ah$, and at the same time

$$\text{ıf } b \in \overline{N}(X), \text{ then } bN \cap X \neq \varnothing .$$

Thus, since $b, h \in \overline{N}(X)$ we have

$$bh^{-1} \in \overline{N}(X) \Rightarrow \left(bh^{-1}\right) N \cap X \neq \varnothing,$$

and so

$$aN \cap X \neq \varnothing \Rightarrow a \in \overline{N}(X) \Rightarrow \left(\overline{N}(X)\right)^{-} \subseteq \overline{N}(X).$$

Hence

$$\overline{N}(X) = \left(\overline{N}(X)\right)^{-}$$

This prove $\overline{N}(X)$ is closed.

Proposition 4.7 *Let N be a normal subgroups of a topological group G and X be a subgroup of G. Then*

(1) If $\underline{N}(X)$ is a topological subgroup of G, then the topological closure of $\underline{N}(X)$, $(\underline{N}(X))^{-}$, is a topological subgroup of G,

(2) If $\overline{N}(X)$ is a topological subgroup of G, then the topological closure of $\overline{N}(X)$, $\left(\overline{N}(X)\right)^{-}$, is a topological subgroup of G.

Proof: (1) Let $\underline{N}(X)$ is a topological subgroup of G. Suppose that $x, y \in (\underline{N}(X))^{-}$. We then prove that $xy \in (\underline{N}(X))^{-}$ and $x^{-1} \in (\underline{N}(X))^{-}$. Let W be a neighborhood of the element xy. Then there exist neighborhoods U and V of the elements x and y such that $UV \subseteq W$. Since $x \in (\underline{N}(X))^{-}$ and $y \in (\underline{N}(X))^{-}$ there exist elements a and b of $\underline{N}(X)$ such that

$$a \in U \cap \underline{N}(X) \text{ and } b \in V \cap \underline{N}(X).$$

Thus we have

$$a \in U, \; b \in V, \; aN \subseteq X, \text{ and } bN \subseteq X.$$

This implies that

$$ab \in UV \text{ and } (ab)\,N \subseteq X.$$

Therefore $UV \cap \underline{N}(X) \neq \varnothing$ and so $W \cap \underline{N}(X) \neq \varnothing$. Hence $xy \in (\underline{N}(X))^{-}$.

Let $x \in (\underline{N}(X))^{-}$. Let W be a neighborhood of the element x^{-1}. Then there exist neighborhoods U of the element x such that $U^{-1} \subseteq W$, where $U^{-1} = \{x^{-1} \mid x \in U\}$. Since $x \in (\underline{N}(X))^{-}$ there exist elements a of $\underline{N}(X)$ such that $a \in U$. Thus $aN \subseteq X$ and $a^{-1} \in U^{-1}$. This implies that $a^{-1}N \subseteq X$, and so $a^{-1} \in U^{-1} \cap \underline{N}(X)$. Thus we have

$$U^{-1} \cap \underline{N}(X) \neq \varnothing, \text{ and so } W \cap \underline{N}(X) \neq \varnothing.$$

Hence $x^{-1} \in (\underline{N}(X))^{-}$. This prove $(\underline{N}(X))^{-}$ is a topological rough subgroup of G.

(2) Let $\overline{N}(X)$ is a topological subgroup of G. Suppose that $x, y \in \left(\overline{N}(X)\right)^{-}$. We then prove that $xy \in \left(\overline{N}(X)\right)^{-}$ and $x^{-1} \in \left(\overline{N}(X)\right)^{-}$. Let W be a neighborhood of the element xy. Then there exist neighborhoods U and V of the elements x and y such that $UV \subseteq W$. Since $x \in \left(\overline{N}(X)\right)^{-}$ and $y \in \left(\overline{N}(X)\right)^{-}$ there exist elements a and b of $\overline{N}(X)$ such that $a \in U$ ve $b \in V$.

Thus we have

$$aN \cap X \neq \varnothing \text{ and } bN \cap X \neq \varnothing.$$

This implies that $(ab)N \cap X \neq \varnothing$, and so $ab \in \overline{N}(X)$. Thus, we have $UV \cap \overline{N}(X) \neq \varnothing$, and so $W \cap \overline{N}(X) \neq \varnothing$. Hence $xy \in \left(\overline{N}(X)\right)^{-}$.

Let $x \in \left(\overline{N}(X)\right)^{-}$. Let W be a neighborhood of the element x^{-1}. Then there exist neighborhoods U of the element x such that $U^{-1} \subseteq W$, where $U^{-1} = \{x^{-1} \mid x \in U\}$. Since $x \in \left(\overline{N}(X)\right)^{-}$ there exist elements a of $\overline{N}(X)$ such that $a \in U$. Thus $a^{-1} \in \overline{N}(X)$ and $a^{-1} \in U^{-1}$. This implies that $a^{-1}N \subseteq X$, and so $a^{-1} \in U^{-1} \cap \overline{N}(X)$. Thus we have

$$U^{-1} \cap \overline{N}(X) \neq \varnothing, \text{ and so } W \cap \overline{N}(X) \neq \varnothing.$$

Hence $x^{-1} \in \left(\overline{N}(X)\right)^{-}$.

This prove $\left(\overline{N}(X)\right)^{-}$ is a topological rough subgroup of G.

In [28] and [11] prove the next two propositions for the lower and upper approximations of a subgroup in a group. In the following we rewrite them in topological group.

Proposition 4.8 *Let N and H be normal subgroups of a topological group G. If X is a subgroup of G, then*

$$\overline{H}(X)\,\overline{N}(X) = \left(\overline{HN}\right)(X).$$

Proof: The proof of the Proposition $\overline{H}(X)\,\overline{N}(X) \subseteq \left(\overline{HN}\right)(X)$ is similar to Proposition 3.5. of [11] and the proof of the Proposition $\overline{H}(X)\,\overline{N}(X) \supseteq \left(\overline{HN}\right)(X)$ is similar to Proposition 3.2. of [28].

Proposition 4.9 *Let N and H be normal subgroups of a topological group G. If X is a subgroup of G, then*

$$\underline{H}(X)\,\underline{N}(X) = (\underline{HN})(X).$$

Proof: The proof is similar to Proposition 3.6. of [28]

5 Homomorphisms and Isomorphisms of Rough Approximations and Subgroups

In this section we mainly study the image and inverse image of rough approximations of a topological subgroup with respect to a continious homomorphism between two topological groups. It is proven that the lower and upper approximations of a topological subgroup are invariant under a continious group homomorphism.

Proposition 5.1 *Let G_1 and G_2 be two topological groups. Let $f : G_1 \to G_2$ be an onto open topological homomorphism, with N a normal subgroup of G_1 and X be any nonempty subset of G_1. Then*

If $X \subseteq N$ and $Ker f = N$, then the quotient group $G_1 / \overline{N}(X)$ is topological isomorphic to G_2.

Proof: Since $X \subseteq N$ and by Proposition 3.5 we have $\overline{N}(X) = N$. Thus the open homomorphism f with $Ker f = \overline{N}(X)$ and Theorem 2.9 we can conclude that $G_1 / \overline{N}(X)$ is topological isomorphic to G_2.

Proposition 5.2 *Let G_1 and G_2 be two topological groups. Let $f : G_1 \to G_2$ be an onto open topological homomorphism, with N a normal subgroup of G_1 and X a subgroup of G_1. Then*

If $N \subseteq X$ and $Ker f = X$, then the quotient group $G_1 / \underline{N}(X)$ is topological isomorphic to G_2.

Proof: By Proposition 3.5 we can get $\underline{N}(X) = X$. Thus the open homomorphism f with $Ker f = \underline{N}(X)$ and Theorem 2.9 we can conclude that $G_1 / \underline{N}(X)$ is topological isomorphic to G_2.

Proposition 5.3 *Let N be a normal subgroups of a topological group G and H a subgroup of G. If $f : H \to \overline{N}(H) / N, x \mapsto xN$, an onto open topological homomorphism, then the quotient group $H / H \cap N$ is topological isomorphic to $\overline{N}(H) / N$.*

Proof: By Proposition 3.5 we have $\overline{N}(H) = HN$, and the Second Isomorphism Theorem, then quotient groups $H / H \cap N$ and $\overline{N}(H) / N$ are isomorphic. Thus the open homomorphism f with $Ker f = H \cap N$ Theorem 2.9 we can conclude that $H / H \cap N$ is topological isomorphic to $\overline{N}(H) / N$.

Proposition 5.4 *Let N and H be normal subgroups of a topological group G and let $\rho : G \to G / N$ be the canonical homomorphism. Then*

(1) If $f : G/N \to G/\overline{H}(N)$, $xN \mapsto x\left(\overline{H}(N)\right)$ and $g : G/H \to G/\overline{N}(H)$, $xH \mapsto x\overline{N}(H)$ are an open map, then the quotient group $(G/N)/\left(\overline{H}(N)/N\right)$ is topological isomorphic to $(G/H)/\left(\overline{N}(H)/H\right)$,

(2) If $N \subseteq H$ and $f : G/N \to G/\underline{N}(H)$, $xN \mapsto x\underline{N}(H)$, is an open map, then the quotient group $G/\underline{N}(H)$ is topological isomorphic to $(G/N)/(\underline{N}(H)/N)$.

Proof: (1) Since N and H are normal subgroups of G, by Proposition 4.3 it follows that $\overline{N}(H)$ and $\overline{H}(N)$ are normal subgroups of G, respectively. By the Third Isomorphism Theorem we have

$$G/\overline{H}(N) \cong (G/N)/\left(\overline{H}(N)/N\right) \text{ ve } G/\overline{N}(H) \cong (G/H)/\left(\overline{N}(H)/H\right).$$

Define $f : G/N \to G/\overline{H}(N)$ by $xN \mapsto x\left(\overline{H}(N)\right)$. Since the maps $\rho : G \to G/N$ and $\varphi : G \to G/\overline{H}(N)$ are an onto open topological homomorhips, then f is an onto open topological homomorhism. Thus the open homomorphism f with $kerf = \overline{H}(N)/N$ and Theorem 2.9 we can conclude that $G/\overline{H}(N)$ is topological isomorphic to $(G/N)/\left(\overline{H}(N)/N\right)$. Similarly, $G/\overline{N}(H)$ is topological isomorphic to $(G/H)/\left(\overline{N}(H)/H\right)$.

It follows from Proposition 3.6 that $\overline{N}(H)=\overline{H}(N)$. Hence

$$(G/N)/\left(\overline{H}(N)/N\right) \cong (G/H)/\left(\overline{N}(H)/H\right).$$

(2) From Proposition 3.5, and the Third Isomorphism Theorem, then

$$G/\underline{N}(H) \cong (G/N)/(\underline{N}(H)/N)$$

Define $f : G/N \to G/\underline{N}(H)$ by $xN \mapsto x\underline{N}(H)$. Since the maps $\rho : G \to G/N$ and $\varphi : G \to G/\underline{N}(H)$ are an onto open topological homomorhips, then f is an onto open topological homomorhism. Thus the open homomorphism f with $Kerf = \underline{N}(H)/N$ Theorem 2.9 we can conclude that $G/\underline{N}(H)$ is topological isomorphic to $(G/N)/(\underline{N}(H)/N)$.

Proposition 5.5 *Let G_1 and G_2 be two topological groups. Let $f : G_1 \to G_2$ be an onto topological homomorphism, with N a normal subgroup of G_1 and X a subgroup of G_1. Then*

(1) If $N \subseteq X$, then $f(\underline{N}(X)) = \underline{f(N)}(f(X))$,

(2) $f\left(\overline{N}(X)\right) = \overline{f(N)}(f(X))$,

(3) If $N \subseteq X$, then $f(X)$ is a lower topological rough subgroup of G_2,

(4) $f(X)$ is a upper topological rough subgroup of G_2.

Proof: (1) Since f is an onto homomorphism from G_1 to G_2, N a normal subgroup of G_1 and X a subgroup of G_1, then $f(N)$ and $f(X)$ are a normal and a subgroup of G_2 respectively.Since $N \subseteq X \Leftrightarrow f(N) \subseteq f(X)$, by Proposition 3.5 we can get $\underline{N}(X) = X \Leftrightarrow \underline{f(N)}(f(X)) = f(X)$.Thus $f(\underline{N}(X)) = \underline{f(N)}(f(X))$.

(2) For any element $a \in \overline{N}(X)$, by the definition of $\overline{N}(X)$ we have $aN \cap X \neq \varnothing$. Since $f(aN \cap X) \subseteq f(aN) \cap f(X)$, it follows that $f(aN \cap X) \subseteq f(aN) \cap f(X) \neq \varnothing$, which implies $f(a) f(N) \cap f(X) \neq \varnothing$. Thus $f(a) \subseteq \overline{f(N)}(f(X))$. Hence $f(\overline{N}(X)) \subseteq \overline{f(N)}(f(X))$.

On the other hand, for any element $y \in \overline{f(N)}(f(X))$, by the definition of $\overline{f(N)}(f(X))$ we have $y\ f(N) \cap (f(X))$. Thus there exist $n \in N$ and $a \in X$ such that $y\ f(n) = f(a)$. Since N is a normal subgroup of G_1, then $f(N)$ is also a normal subgroup of G_2. Thus $f(n)^{-1} = f(N)$. Hence $y = f(a) f(n)^{-1} = f(a) f(n^{-1}) = f(an^{-1})$. Since $a = (an^{-1}) n \in an^{-1}N \cap X \neq \varnothing$, it follows that $an^{-1} \in \overline{N}(X)$. This means $y \in f(\overline{N}(X))$. Thus $\overline{f(N)}(f(X)) \subseteq f(\overline{N}(X))$.Hence $f(\overline{N}(X)) = \overline{f(N)}(f(X))$.

(3) Let $N \subseteq X$. Since f is an onto homomorphism from G_1 to G_2, N a normal subgroup of G_1 and X a subgroup of G_1, then $f(N)$ and $f(X)$ are a normal and a subgroup of G_2, respectively. Since $N \subseteq X \Leftrightarrow f(N) \subseteq f(X)$, by item (1) and Proposition 4.4 we have $f(X)$ is a lower topological rough subgroup of G_2.

(4) By item (2) and Proposition 4.2 we have $f(X)$ is a upper topological rough subgroup of G_2.

Proposition 5.6 *Let G_1 and G_2 be two groups. Let $f : G_1 \to G_2$ be an onto topological homomorphism, with N_2 a normal subgroup of G_2 and X a nonempty subset of G_2. Then*

(1) If $N_2 \subseteq X$, then $f^{-1}(\underline{N_2}(X)) = \underline{f^{-1}(N_2)}(f^{-1}(X))$,

(2) $f^{-1}(\overline{N_2}(X)) = \overline{f^{-1}(N_2)}(f^{-1}(X))$,

(3) If $N_2 \subseteq X$, then $f^{-1}(X)$ is a lower topological rough subgroup of G_1,

(4) $f^{-1}(X)$ is a upper topological rough subgroup of G_1.

Proof: Similar to the proof of Proposition 5.5.

Proposition 5.7 *Let G_1 and G_2 be two groups. Let $f : G_1 \to G_2$ be an onto topological homomorphism, and N a normal subgroup of G_1. Let X be any normal subgroup of G_1 and $Kerf \subseteq N$. Then*

(1) If f open map, then the quotient group $G_1 \diagup \overline{N}(X)$ is topological isomorphic to $G_2 \diagup \overline{f(N)}(f(X))$,

(2) If N closed, then $f\left(\overline{N}(X)\right)$ is a closed topological subgroup of G_2,

(3) If f open map and $N \subseteq X$, then the quotient group $G_1 \diagup \underline{N}(X)$ is topological isomorphic to $G_2 \diagup \underline{f(N)}(f(X))$,

(4) If $N \subseteq X$ and N closed, then $f(\underline{N}(X))$ is a closed topological subgroup of G_2.

Proof: (1) From Proposition 5.5, we have $f\left(\overline{N}(X)\right) = \overline{f(N)}(f(X))$. Since the complete inverse image of $f\left(\overline{N}(X)\right)$ is $f^{-1}\left(f\left(\overline{N}(X)\right)\right) = \overline{N}(X)\,Kerf$ we only need to prove $\overline{N}(X)\,Kerf = \overline{N}(X)$.

Since N and X are normal subgroups of G_1, it follows from Proposition 4.3 that $\overline{N}(X)$ is a normal subgroup of G_1. This means the identity element

$$e \in \overline{N}(X) \Rightarrow eN \cap X \neq \varnothing \Rightarrow N \cap X \neq \varnothing.$$

Thus we have $nN \cap X = N \cap X \neq \varnothing$ for any $n \in N$. Hence $n \in \overline{N}(X)$ and so $N \subseteq \overline{N}(X)$. Since, $Kerf \subseteq N$,then $Kerf \subseteq N \subseteq \overline{N}(X)$. Thus $\overline{N}(X)\,Kerf$ $=\overline{N}(X)$. Since f is open map, by Theorem 2.10 we have $G_1 \diagup \overline{N}(X)$ is topological isomorphic to $G_2 \diagup \overline{f(N)}(f(X))$.

(2) Let N closed. Then from proposition 3.3 we have $\overline{N}(X)$ closed. By the above proof we have $f^{-1}\left(f\left(\overline{N}(X)\right)\right) = \overline{N}(X)$. Since f continuous, then $f\left(\overline{N}(X)\right)$ closed. On the other hand, by proposition 5.5 item (4) we have $f\left(\overline{N}(X)\right)$ is a topological subgroup of G_2. Hence $f\left(\overline{N}(X)\right)$ is a closed topological subgroup of G_2

(3) Let $Çekf \subseteq N \subseteq X$. Then from Proposition 3.5 we have $\underline{N}(X) = X$ and $\underline{f(N)}(f(X)) = f(X)$. Since the complete inverse image of $f(X)$ is $f^{-1}(f(X)) = KerfX = X$, and so f is open map, by Theorem 2.10 we have $G_1 \diagup \underline{N}(X)$ is topological isomorphic to $G_2 \diagup \underline{f(N)}(f(X))$.

(4) Let $N \subseteq X$ and N closed. Then from Proposition 3.3 that $\underline{N}(X)$ closed. By the above proof we have $f^{-1}(f(\underline{N}(X))) = \underline{N}(X)$.Since f continuous, then $f(\underline{N}(X))$ closed. On the other hand, by Proposition 5.5 item (3) we have $f(\underline{N}(X))$ is a topological subgroup of G_2. Hence $f(\underline{N}(X))$ is a closed topological subgroup of G_2.

6 Conclusion

In this paper we deal with one of the newest argument from rough set theory namely topological rough approximations and subgroups in topological groups. Then, we present the concept of topological rough subgroups and prove some properties. Finally, the topological isomorphism theorems of topological rough approximations and subgroups are proved.

References

[1] A.A. Allam, M.Y. Bakeir and E.A. Abo-Tabl, Some methods for generating topologies by relations, *Bull. Malays. Math. Sci. Soc.*, 31(2)(1) (2008), 35-45.

[2] R. Biswas and S. Nanda, Rough groups and rough subgroups, *Bull. Polish Acad. Sci. Math*, 42(1994), 251-254.

[3] Z. Bonikowaski, Algebraic structures of rough sets, In: W.P. Ziarko (Ed.), *Rough Sets, Fuzzy Sets and Knowledge Discovery*, Springer-Verlag, Berlin, (1995), 242-247.

[4] W. Cheng, Z. Mo and J. Wang, Notes on the lower and upper approximations in a fuzzy group and rough ideals in semigroups, *Inform. Sci.* 177(2007), 5134-5140.

[5] B. Davvaz, Roughness in rings, *Inform. Sci.*, 164(2004), 147-163.

[6] B. Davvaz and M. Mahdavipour, Roughness in modules, *Inform. Sci.*, 176(2006), 3658-3674.

[7] T. Iwinski, Algebraic approach to rough sets, *Bull. Polish Acad. Sci. Math.*, 35(1987), 673-683.

[8] J. Jarvinen, Properties of rough approximations, *J. Adv. Comput. Intel. and Intelligent Inform.*, 9(5) (2005), 502-505.

[9] J. Jarvinen, On the structure of rough approximations, *Fund. Inform.*, 53(2002), 135-153.

[10] M. Kondo, On the structure of generalized rough sets, *Inform. Sci.*, 176(2006), 589-600.

[11] N. Kuroki and P.P. Wang, The lower and upper approximations in a fuzzy group, *Information Sciences*, 90(1996), 203-220.

[12] N. Kuroki, Rough ideals in semigroups, *Inform. Sci.*, 100(1997), 139-163.

[13] E.F. Lashin et al, Rough set theory for topological spaces, *Int. J. Aprrox. Reason.*, 40(2005), 35-43.

[14] Z. Li, T. Xie and Q. Li, Topological structure of generalized rough sets, *Comput. Math. with Appl.*, 63(2012), 1066-1071.

[15] F. Li and Z. Zhang, The homomorphisms and operations of rough groups, *The Scientific World Journal*, 2014(2014), 1-7.

[16] D. Miao, S. Han, D. Li and L. Sun, *Rough Group, Rough Subgroup and Their Properties*, D. Ślezak et al. (Eds.), RSFDGrC, LNAI 3641, Springer-Verlag Berlin, Heidelberg, (2005), 104-113.

[17] A.F. Özcan, N. Bağırmaz, H. Taşbozan and İ. İçen, Topologies and approximation operators induced by binary relations, *IECMSA-2013*, Sarajevo, Bosnıa and Herzegovina, August (2013).

[18] Z. Pawlak, Rough sets, *Int. J. Comput. Inform. Sci.*, 11(1982), 341-356.

[19] Z. Pawlak, Rough sets and intelligent data analysis, *Inform. Sci.*, 147(2002), 1-12.

[20] Z. Pawlak and A. Skowron, Rudiments of rough sets, *Inform. Sci.*, 177(2007), 3-27.

[21] Z. Pei, D. Pei and L. Zheng, Topology vs generalized rough sets, *Int. J. Aprrox. Reason.*, 52(2011), 231-239.

[22] L. Polkowski and A. Skowron (Eds.), *Rough Sets and Current Trends in Computing (vol. 1424)*, Springer, Berlin, Germany, (1998).

[23] J. Pomykala and J.A. Pomykala, The stone algebra of rough sets, *Bull. Polish Acad. Sci. Math.*, 36(1998), 495-508.

[24] L.S. Pontryagin, *Topological Groups*, Princeton Univ. Press, (1958) (Translated from Russian).

[25] A.S. Salama and M.M.E.A. El-Monsef, New topological approach of rough set generalizations, *Internat. J. Computer Math.*, 8(7) (2011), 1347-1357.

[26] A. Skowron, On the topology in information systems, *Bull. Polish Acad. Sci. Math.*, 36(1988), 477-480.

[27] M.L. Thivagar, C. Richard and N.R. Paul, Mathematical innovations of a modern topology in medical events, *Internat. J. Inform. Sci.*, 2(4) (2012), 33-36.

[28] C.Z. Wang and D.G. Chen, A short note on some properties of rough groups, *Comput. Math. Appl.*, 59(2010), 431-436.

[29] Z. Wang and L. Shu, The lower and upper approximations in a group, *International Journal of Mathematical and Computational Sciences*, 6(2012), 158-162.

[30] C. Wang, D. Chen and Q. Hu, On rough approximations of groups, *Int. J. Mach. Learn. & Cyber*, Springer-Verlag, 4(2013), 445-449.

[31] M. Vlach, Algebraic and topological aspects of rough set theory, *Fourth International Workshop on Computational Intelligence Applications*, IEEE SCM Hiroshima Chapter, Hiroshima University, Japan, December (2008).

[32] A. Wiweger, On topological rough sets, *Bull. Polish Acad. Sci. Math.*, 37(1988), 51-62.

[33] Y.Y. Yao, Constructive and algebraic methods of the theory of rough sets, *Inform. Sci.*, 109(1998), 21-47.