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Homoderivations on Rings

Asmaa Melaibari[1], Najat Muthana[2] and Ahmad Al-Kenani[3]

[1,2,3]Department of Mathematics
King Abdulaziz University, Jeddah, Saudi Arabia
[1]E-mail: melaibari@gmail.com
[2]E-mail: nmuthana@kau.edu.sa
[3]E-mail: analkenani@kau.edu.sa

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Abstract

Let R be a ring with center $Z(R)$ and let U be a nonzero ideal. An additive mapping $h : R \to R$ is called a homoderivation on R if $h(xy) = h(x)h(y) + h(x)y + xh(y)$ for all $x, y \in R$. In this paper, we prove the commutativity of R if any of the following conditions is satisfied: (i) $[x, y] = [h(x), h(y)]$ for all $x, y \in R$, (ii) $h([x, y]) = 0$ for all $x, y \in U$, and (iii) $h([x, y]) \in Z(R)$ for all $x, y \in R$.

Keywords: *Prime rings, Semiprime rings, Strong commutativity-preserving mappings, Homoderivations, Commutativity theorems.*

1 Introduction

Let R be a ring with center $Z(R)$. A derivation on R is an additive mapping $d : R \to R$ satisfying $d(xy) = d(x)y+xd(y)$ for all $x, y \in R$. In 2000, El Sofy [4] defined a homoderivation on R as an additive mapping $h : R \to R$ satisfying $h(xy) = h(x)h(y)+h(x)y+xh(y)$ for all $x, y \in R$. An example of such mapping is to let $h(x) = f(x) - x$ for all $x \in R$ where f is an endomorphism on R. It is clear that a homoderivation h is also a derivation if $h(x)h(y) = 0$ for all $x, y \in R$. In this case, $h(x)Rh(y) = 0$ for all $x, y \in R$. So, if R is a prime ring, then the only additive mapping which is both a derivation and a homoderivation is the zero mapping.

If $S \subseteq R$, then a mapping $f : R \to R$ preserves S if $f(S) \subseteq S$. A mapping $f : R \to R$ is zero-power valued on S if f preserves S and if for each $x \in S$, there exists a positive integer $n(x) > 1$ such that $f^{n(x)}(x) = 0$. A mapping $f : R \to R$ is strong commutativity-preserving (scp) on S if $[x, y] = [f(x), f(y)]$ for all $x, y \in S$.

In this paper, we prove commutativity theorems analogous to some of the results presented in [1, 2, 3] using the concept of homoderivations.

2 On the SCP Condition

Bell and Daif [2] proved the commutativity of semiprime rings admitting strong commutativity-preserving derivations. Our purpose in this section is to prove a similar result regarding homoderivations. To do this, we will use the following lemma.

Lemma 2.1 ([2], Lemma 1(a))**.** *If R is a semiprime ring, then the center of a nonzero one-sided ideal is contained in the center of R; in particular, any commutative one-sided ideal is contained in the center of R.*

Theorem 2.2. *Let R be a semiprime ring, U a nonzero ideal, and h a homoderivation on R which is zero-power valued on U. If h is strong commutativity-preserving on U, then $U \subseteq Z(R)$.*

Proof: By hypothesis, we have

$$[x, y] = [h(x), h(y)] \qquad \text{for all} \quad x, y \in U. \tag{1}$$

Replacing y by yx, we get $[x, yx] = [h(x), h(yx)]$. Therefore,

$$\begin{aligned} [x, y]x &= [h(x), h(y)h(x) + h(y)x + yh(x)] \\ &= [h(x), h(y)h(x)] + [h(x), h(y)x] + [h(x), yh(x)] \\ &= [h(x), h(y)]h(x) + h(y)[h(x), x] + [h(x), h(y)]x + [h(x), y]h(x). \end{aligned}$$

Using (1), we get

$$[x, y]h(x) + h(y)[h(x), x] + [h(x), y]h(x) = 0$$

which is equivalent to

$$[x + h(x), y]h(x) + h(y)[h(x), x] = 0 \qquad \text{for all} \quad x, y \in U. \tag{2}$$

Replacing y by $yh(x)$ in (1), we get $[x, yh(x)] = [h(x), h(yh(x))]$. Therefore,

$$\begin{aligned} y[x, h(x)] + [x, y]h(x) &= [h(x), h(y)h^2(x) + h(y)h(x) + yh^2(x)] \\ &= [h(x), h(y)h^2(x)] + [h(x), h(y)h(x)] + [h(x), yh^2(x)] \\ &= h(y)[h(x), h^2(x)] + [h(x), h(y)]h^2(x) + [h(x), h(y)]h(x) \\ &\quad + y[h(x), h^2(x)] + [h(x), y]h^2(x). \end{aligned}$$

Using (1), we get

$$h(y)[x,h(x)]+[x,y]h^2(x)+[h(x),y]h^2(x)=0$$

which is equivalent to

$$h(y)[x,h(x)]+[x+h(x),y]h^2(x)=0 \qquad \text{for all} \ \ x,y\in U. \tag{3}$$

Adding (2) and (3), we obtain

$$[x+h(x),y]h(x+h(x))=0 \qquad \text{for all} \ \ x,y\in U. \tag{4}$$

Since h is zero-power valued on U, there exists an integer $n(x)>1$ such that $h^{n(x)}(x)=0$ for all $x\in U$. Replacing x by $x-h(x)+h^2(x)+\cdots+(-1)^{n(x)-1}h^{n(x)-1}(x)$ in (4), we get

$$[x,y]h(x)=0 \qquad \text{for all} \ \ x,y\in U.$$

Replacing y by yw where $w\in U$ and simplifying, we get

$$[x,y]wh(x)=0 \qquad \text{for all} \ \ w,x,y\in U.$$

Therefore,

$$[x,y]Uh(x)=0 \qquad \text{for all} \ \ x,y\in U.$$

Since U is an ideal, the above yields that

$$[x,y]URh(x)=0 \qquad \text{for all} \ \ x,y\in U. \tag{5}$$

Let P_α be a family of prime ideals of R such that $\bigcap P_\alpha=\{0\}$, and let P denote a typical P_α. From (5) it follows that for each $x\in U$, we have either

$$[x,y]U\subseteq P \qquad \text{or} \qquad h(x)\in P. \tag{6}$$

If $h(x)\in P$, then clearly, $[h(x),h(y)]\in P$ for all $y\in U$. Hence, by (1), $[x,y]\in P$ for all $y\in U$. And so we have $[x,y]U\subseteq P$ for all $y\in U$.
Hence, both conditions in (6) imply that $[x,y]U\subseteq P$ for all $x,y\in U$ and $P\in P_\alpha$. Since $\bigcap P_\alpha=\{0\}$, $[x,y]U=0$ for all $x,y\in U$. Since U is an ideal, we have $[x,y]RU=0$ and thus $[x,y]R[x,y]=0$ for all $x,y\in U$. Since R is semiprime, it follows that $[x,y]=0$ for all $x,y\in U$. Hence, U centralizes U. By Lemma 2.1, $U\subseteq Z(R)$.

Corollary 2.3. *If R is a semiprime ring admitting a zero-power valued homoderivation which is strong commutativity- preserving on R, then R is commutative.*

3 On the Condition $h([x,y])=0$

Bell and Daif [3] studied the commutativity of prime rings admitting a derivation d that satisfies $d([x,y])=0$ on two-sided ideals (or one-sided ideals). Motivated by their results, we investigate the commutativity of rings admitting a homoderivation h such that $h([x,y])=0$. We begin with the following useful lemma.

Lemma 3.1 ([2], Lemma 1(b)). *If R is a prime ring, then the centralizer of any one-sided ideal is equal to the center of R. Thus, if R has a nonzero central right ideal, then R must be commutative.*

Theorem 3.2. *Let R be a prime ring and U a nonzero ideal of R. If R admits a nonzero homoderivation h that satisfies $h([x,y])=0$ for all $x,y\in U$, then R is commutative.*

Proof: By hypothesis, we have

$$h([x,y])=0 \qquad \text{for all } \ x,y\in U. \tag{7}$$

Replacing y by yx, we get $0=h([x,yx])=h([x,y]x)=h([x,y])h(x)+h([x,y])x+[x,y]h(x)$ for all $x,y\in U$. Applying (7), we get

$$[x,y]h(x)=0 \qquad \text{for all } \ x,y\in U. \tag{8}$$

Replacing y by ry for arbitrary $r\in R$ gives $0=[x,ry]h(x)=r[x,y]h(x)+[x,r]yh(x)$ for all $x,y\in U$. It follows by (8) that

$$[x,r]yh(x)=0 \qquad \text{for all } \ x,y\in U, r\in R.$$

Therefore, $[x,r]Uh(x)=0$ for all $x\in U$ and $r\in R$. Since U is an ideal, we have $[x,r]RUh(x)=0$ for all $x\in U$ and $r\in R$. Since R is prime, we have for each $x\in U$ either

$$x\in Z(R) \qquad \text{or} \qquad Uh(x)=0.$$

The sets of $x\in U$ for which these two conditions hold are additive subgroups of U whose union is U. But since a group cannot be the union of two of its proper subgroups, we find that

$$U\subseteq Z(R) \qquad \text{or} \qquad Uh(U)=0.$$

If $Uh(U)=0$, then $xh(y)=0$ for all $x,y\in U$. Replacing y by yr for arbitrary $r\in R$, we get $0=xh(yr)=xh(y)h(r)+xh(y)r+xyh(r)=xyh(r)$ for all $x,y\in U$. Therefore, $U^2h(r)=0$ for all $r\in R$. Since h is not zero, we must have $h(r_0)\neq 0$ for some $r_0\in R$. By primeness of R, $U^2=0$ which implies that $U=0$ which is a contradiction. Hence, we must have $U\subseteq Z(R)$. By Lemma 3.1, R is commutative.

Next, we generalize the last result to right ideals.

Theorem 3.3. *Let R be a prime ring with characteristic different from 2 and let U be a nonzero right ideal of R. If h is a nonzero homoderivation on R which preserves U and satisfies $h([x,y]) = 0$ for all $x, y \in U$, then either R is commutative or $h^2(U) = 0 = h(U)h(U)$.*

Proof: By hypothesis, we have

$$h([x,y]) = 0 \qquad \text{for all } \ x, y \in U. \tag{9}$$

Expanding this expression, we find that

$$[h(x), h(y)] + [h(x), y] + [x, h(y)] = 0 \qquad \text{for all } \ x, y \in U. \tag{10}$$

Replacing x by x^2, we get

$$[h(x^2), h(y)] + [h(x^2), y] + [x^2, h(y)] = 0 \qquad \text{for all } \ x, y \in U. \tag{11}$$

Expanding (11) and using (10) yields

$$h(x)[x,y] + [x,y]h(x) = 0 \qquad \text{for all } \ x, y \in U. \tag{12}$$

Replacing y by $[x,y]$ in (10) and using (9), we get

$$h(x)[x,y] - [x,y]h(x) = 0 \qquad \text{for all } \ x, y \in U. \tag{13}$$

Comparing (12) and (13), it follows that

$$2h(x)[x,y] = 0 \qquad \text{for all } \ x, y \in U.$$

Since char $R \neq 2$, we find that

$$h(x)[x,y] = 0 \qquad \text{for all } \ x, y \in U.$$

Replacing y by yw where $w \in U$, we get $h(x)[x, yw] = 0$. Thus,

$$h(x)y[x,w] = 0 \qquad \text{for all } \ w, x, y \in U.$$

Since U is a right ideal, the above yields that

$$h(x)UR[x,w] = 0 \qquad \text{for all } \ w, x \in U.$$

Since R is prime, for each $x \in U$, we have either $h(x)U = 0$ or x centralizes U. By Lemma 3.1, we can say that for each $x \in U$,

$$h(x)U = 0 \qquad \text{or} \qquad x \in Z(R).$$

The sets of $x \in U$ for which these two conditions hold are additive subgroups of U whose union is U. But since a group cannot be the union of two of its proper subgroups, we find that

$$h(U)U = 0 \qquad \text{or} \qquad U \subseteq Z(R).$$

Assume that R is not commutative. Then, by Lemma 3.1, U is not a central ideal. Thus, $h(U)U = 0$; and in particular, $h(U)h(U) = 0$ since $h(U) \subseteq U$. Substituting these conditions in (10) yields

$$xh(y) = yh(x) \qquad \text{for all} \;\; x, y \in U.$$

Replacing y by $yh(x)$, we get $xh(yh(x)) = yh(x)h(x) = 0$. Therefore,

$$xyh^2(x) = 0 \qquad \text{for all} \;\; x, y \in U.$$

Since U is a right ideal, the above yields that

$$xURh^2(x) = 0 \qquad \text{for all} \;\; x \in U.$$

Since R is prime, for each $x \in U$, we have either

$$xU = 0 \qquad \text{or} \qquad h^2(x) = 0.$$

With the same additive subgroups argument, it follows that

$$U^2 = 0 \qquad \text{or} \qquad h^2(U) = 0.$$

But since R is prime and U is a nonzero right ideal, $U^2 \neq 0$ and hence $h^2(U) = 0 = h(U)h(U)$.

4 On the Condition $h([x, y]) \in Z(R)$

Bell [1] proved that if R is a prime ring of characteristic different from 2 and d is a nonzero derivation on R satisfying $d([x, y]) \in Z(R)$ for all $x, y \in R$, then R is commutative. In this section, we prove a similar result using the concept of homoderivations. This result is an extension to Theorem 3.2. We first need to prove the following lemma.

Lemma 4.1. *Let R be a ring and let h be a zero-power valued homoderivation on R. Then h preserves $Z(R)$.*

Proof: Let $x \in R$ be arbitrary. Then, for $z \in Z(R)$, we have

$$h(xz) = h(x)h(z) + h(x)z + xh(z), \qquad \text{and}$$
$$h(zx) = h(z)h(x) + h(z)x + zh(x).$$

Subtracting the two equations and using the fact that $z \in Z(R)$ yield

$$[h(x)+x, h(z)] = 0 \qquad \text{for all } \; x \in R \; \text{ and } z \in Z(R).$$

Since h is zero-power valued on R, $[x, h(z)] = 0$ for all $x \in R$ and $z \in Z(R)$. This proves that $h(Z(R)) \subseteq Z(R)$.

Lemma 4.2 ([5], Corollary to Lemma 1.1.7)**.** *Let R be a prime ring and suppose that $a \neq 0$ in R satisfies $a[u, x] = 0$ for all $x \in R$. Then $u \in Z(R)$.*

Lemma 4.3 ([1], Lemma 1.3)**.** *If R is a prime ring and $x \in R$ such that $[x, [y, w]] = 0$ for all $w, y \in R$. Then $x \in Z(R)$.*

Lemma 4.4 ([4], Theorem 3.3.3)**.** *Let R be a prime ring, h a nonzero homoderivation of R such that $[h(x), h(y)] = 0$ for all $x, y \in R$. If the characteristic of R is not 2, R is a commutative integral domain.*

Theorem 4.5. *Let R be a prime ring with characteristic different from 2. If $0 \neq h$ is a zero-power valued homoderivation on R such that $h([x, y]) \in Z(R)$ for all $x, y \in R$, then R is commutative.*

Proof: By hypothesis, we have $h([x, y]) \in Z(R)$ for all $x, y \in R$. Therefore,

$$[h(x)+x, h(y)] + [h(x), y] \in Z(R) \qquad \text{for all } \; x, y \in R. \tag{14}$$

Replacing y by zy where $z \in Z(R)$, we obtain

$$[h(x)+x, h(zy)] + [h(x), zy] \in Z(R) \qquad \text{for all } \; x, y \in R.$$

Expanding this condition, we get $h(z)[h(x)+x, h(y)] + [h(x)+x, h(z)]h(y) + h(z)[h(x)+x, y] + [h(x)+x, h(z)]y + z[h(x)+x, h(y)] + [h(x)+x, z]h(y) + z[h(x), y] + [h(x), z]y \in Z(R)$ for all $x, y \in R$ and $z \in Z(R)$. Applying (14) and the fact that $z, h(z) \in Z(R)$, we obtain

$$h(z)[x, y] \in Z(R) \qquad \text{for all } \; x, y \in R, z \in Z(R).$$

In particular,

$$[h(z)[x, y], r] = 0 \qquad \text{for all } \; r, x, y \in R, z \in Z(R). \tag{15}$$

Expanding (15) and using the fact that $h(z) \in Z(R)$, we get

$$h(z)[[x, y], r] = 0 \qquad \text{for all } \; r, x, y \in R, z \in Z(R).$$

If $h(Z(R)) \neq 0$, then there exists $z_0 \in Z(R)$ such that $h(z_0) \neq 0$. By Lemma 4.2, it follows that $[[x, y], r] = 0$ for all $r, x, y \in R$. By Lemma 4.3, $r \in Z(R)$ for all $r \in R$. Thus, in this case, R is commutative.

Now suppose that $h(Z(R)) = 0$. Then by hypothesis, $h^2([x, y]) = 0$ for all $x, y \in R$. Expanding this condition, we get

$$[h^2(x) + 2h(x) + x, h^2(y)] + 2[h^2(x) + h(x), h(y)] + [h^2(x), y] = 0. \qquad (16)$$

Replacing y by $[w, y]$ for arbitrary $w \in R$ and using the facts that $h([x, y]) \in Z(R)$ and $h^2([x, y]) = 0$ for all $x, y \in R$, we obtain

$$[h^2(x), [w, y]] = 0 \qquad \text{for all} \quad w, x, y \in R.$$

Thus, by Lemma 4.3, $h^2(x) \in Z(R)$ for all $x \in R$. Substituting this condition in (16) gives $2[h(x), h(y)] = 0$ for all $x, y \in R$. Since char $R \neq 2$, we arrive at

$$[h(x), h(y)] = 0 \qquad \text{for all} \quad x, y \in R..$$

Hence, by Lemma 4.4, R is commutative.

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