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# Partial Sums of Certain Classes of Analytic Functions Defined by Generalized Differential Operator

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**Abstract**

*In this paper, we introduce the class* $\mathfrak{L}_{\lambda,n}^{\rho,\eta,\mu}(\alpha,\beta)$ *of analytic functions in the open unit disc* $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ *defined by differential operator. The main aim of the present paper is to determine coefficient estimates and some results concerning the partial sums for functions* $f(z)$ *belonging to this class.*

**Keywords:** *Partial sums, Analytic functions, Starlike functions, Convex functions, Differential operator.*

## 1 Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $K(\alpha)$ and $S^*(\alpha)$ denote the subclasses of $A$ which are, respectively, convex and starlike functions of order $\alpha, 0 \leq \alpha < 1$. For convenience, we write $K(0) = K$ and $S^*(0) = S^*$ (see [9]).

For a function $f \in A, \rho, \eta, \mu, \lambda \geq 0$ and $n \in \mathbb{N}_0$ we define the differential operator, as follows:

$$D^0 f(z) = f(z),$$

$$D^1_\lambda(\rho,\eta,\mu)f(z) = (\frac{\rho-\mu+\eta-\lambda}{\rho+\eta})f(z) + (\frac{\mu+\lambda}{\rho+\eta})zf'(z),$$
$$D^2_\lambda(\rho,\eta,\mu)f(z) = D(D^1_\lambda(\rho,\eta,\mu)f(z)),$$
$$.$$
$$.$$
$$.$$
$$D^k_\lambda(\rho,\eta,\mu)f(z) = D(D^{k-1}_\lambda(\rho,\eta,\mu)f(z)). \tag{2}$$

If $f$ is given by (1), then from (2), we see that

$$D^k_\lambda(\rho,\eta,\mu)f(z) = z + \sum_{n=2}^{\infty}(\frac{\rho+(\mu+\lambda)(n-1)+\eta}{\rho+\eta})^k a_n z^n, \tag{3}$$

which generalizes many operators. Indeed, if in the definition of $D^k_\lambda(\rho,\eta,\mu)$ we substitute the following:

- $\eta = 1, \mu = 0$, we get $D^k_\lambda(\rho,1,0)f(z) = D^k f(z) = z + \sum_{n=2}^{\infty}(\frac{\rho+(\lambda(n-1)+1}{\rho+1})^k a_n z^n$ of Aouf, El-Ashwah and El-Deeb differential operator [2].
- $\rho = 1, \eta = 0, \mu = 0$, we get $D^k_\lambda(1,0,0)f(z) = D^k f(z) = z + \sum_{n=2}^{\infty}(1+\lambda(n-1))^k a_n z^n$ of AL-Oboudi differential operator [1].
- $\rho = 1, \eta = 0, \mu = 0$ and $\lambda = 1$, we get $D^k_1(1,0,0)f(z) = D^k f(z) = z + \sum_{n=2}^{\infty}(n)^k a_n z^n$ of Salagean differential operator [6].
- $\rho = 1, \eta = 1, \lambda = 1$ and $\mu = 0$, we get $D^k_1(1,1,0)f(z) = D^k f(z) = z + \sum_{n=2}^{\infty}(\frac{n+1}{2})^k a_n z^n$ of Uralegaddi and Somanatha operator [10].
- $\eta = 1, \lambda = 1$, and $\mu = 0$, we get $D^k_1(\alpha,1,0)f(z) = D^k f(z) = z + \sum_{n=2}^{\infty}(\frac{n+\alpha}{\alpha+1})^k a_n z^n$ of Cho and Srivastava differential operator [3, 4].

Now we define some new subclasses of analytic functions by using extended multiplier transformations operator.

For $0 \le \alpha < 1, \beta \ge 0$ and for all $z \in \mathcal{U}$, let $\mathfrak{A}^{\rho,\eta,\mu}_{\lambda,n}(\alpha,\beta)$ denote the subclass of $A$ consisting of functions $f(z)$ of the form (1) and satisfying the analytic criterion

$$\Re\{\frac{D^k_\lambda(\rho,\eta,\mu)f(z)}{z(D^k_\lambda(\rho,\eta,\mu)f(z))'} - \alpha\} > \beta|\frac{D^k_\lambda(\rho,\eta,\mu)f(z)}{z(D^k_\lambda(\rho,\eta,\mu)f(z))'} - 1|. \tag{4}$$

Denote by $T$ the subclass of $A$ consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n (a_n \geq 0), \tag{5}$$

which are analytic in $\mathcal{U}$. We define the class $\mathfrak{L}_{\lambda,n}^{\rho,\eta,\mu}(\alpha, \beta)$ by:

$$\mathfrak{L}_{\lambda,n}^{\rho,\eta,\mu}(\alpha, \beta) = \mathfrak{A}_{\lambda,n}^{\rho,\eta,\mu}(\alpha, \beta) \cap T.$$

Putting $k = 0$ in (5), the class $\mathfrak{L}_{\lambda,n}^{\rho,\eta,\mu}(\alpha, \beta)$ reduces to the class $ST(\alpha, \beta)$

$$= \{f \in T : \Re\{\frac{f(z)}{zf'(z)} - \alpha\} > \beta|\frac{f(z)}{zf'(z)} - |, 0 \leq \alpha < 1, \beta \geq 0, z \in \mathcal{U}\},$$

and the class $ST(\alpha, 0) = ST(\alpha)$ is the family of functions $f(z) \in T$ which satisfy the following condition (see [5] and [11])

$$ST(\alpha) = \Re\{\frac{f(z)}{zf'(z)}\} > \alpha(0 \leq \alpha < 1).$$

In this paper, applying methods used by Silverman [7] and Silvia [8], we investigate the ratio of a function of the form (5) to its sequence of partial sums $f_m(z) = z + \sum_{n=2}^{m} a_n z^n$. More precisely, we will determine sharp lower bounds for $\Re\{\frac{f(z)}{zf_m(z)}\}, \Re\{\frac{f_m(z)}{f(z)}\}, \Re\{\frac{f'(z)}{f'_m(z)}\}$ and $\Re\{\frac{f'_m(z)}{f'(z)}\}$. In the sequel, we will make use of the well-known result that $\Re\{\frac{1+w(z)}{1-w(z)}\} > 0 (z \in \mathcal{U})$ if and only if $w(z) = \sum_{n=1}^{\infty} c_n z^n$ satisfies the inequality $|w(z)| \leq |z|$.

## 2 Coefficient Estimates

Using the technique used by Yamakawa [11, Lemma 2.1] we prove the following theorem:

**Theorem 2.1** *A function $f(z)$ of the form (1) is in the class $\mathfrak{L}_{\lambda,n}^{\rho,\eta,\mu}(\alpha, \beta)$ if*

$$\sum_{n=2}^{\infty} [2n - n(\alpha - \beta) - (\beta + 1)](\frac{\rho + (\mu + \lambda)(n-1) + \eta}{\rho + \eta})^k a_n \leq 1 - \alpha, \tag{6}$$

*where $-1 \leq \alpha < 1, \beta \geq 0, n \geq 0, \rho, \eta, \mu, \lambda \geq 0$ and $z \in \mathcal{U}$.*

**Proof:** Suppose that (6) is true. Since

$$\frac{[2n - n(\alpha - \beta) - (\beta + 1)]}{1 - \alpha}(\frac{\rho + (\mu + \lambda)(n-1) + \eta}{\rho + \eta})^k - n(\frac{\rho + (\mu + \lambda)(n-1) + \eta}{\rho + \eta})^k$$

$$= \frac{(n-1)(1+\beta)}{1-\alpha}(\frac{\rho+(\mu+\lambda)(n-1)+\eta}{\rho+\eta})^k > 0,$$

we deduce

$$\sum_{n=2}^{\infty} n(\frac{\rho+(\mu+\lambda)(n-1)+\eta}{\rho+\eta})^k |a_n|$$

$$< \sum_{n=2}^{\infty} \frac{[2n-n(\alpha-\beta)-(\beta+1)]}{1-\alpha}(\frac{\rho+(\mu+\lambda)(n-1)+\eta}{\rho+\eta})^k |a_n| \leq 1.$$

It suffices to show that

$$\beta|\frac{D_\lambda^k(\rho,\eta,\mu)f(z)}{z(D_\lambda^k(\rho,\eta,\mu)f(z))'}-1| - \Re(\frac{D_\lambda^k(\rho,\eta,\mu)f(z)}{z(D_\lambda^k(\rho,\eta,\mu)f(z))'}-1) \leq 1-\alpha,$$

we have

$$\beta|\frac{D_\lambda^k(\rho,\eta,\mu)f(z)}{z(D_\lambda^k(\rho,\eta,\mu)f(z))'}-1| - \Re(\frac{D_\lambda^k(\rho,\eta,\mu)f(z)}{z(D_\lambda^k(\rho,\eta,\mu)f(z))'}-1)$$

$$\leq (1+\beta)|\frac{D_\lambda^k(\rho,\eta,\mu)f(z)}{z(D_\lambda^k(\rho,\eta,\mu)f(z))'}-1|$$

$$\leq \frac{(1+\beta)\sum_{n=2}^{\infty}(n-1)(\frac{\rho+(\mu+\lambda)(n-1)+\eta}{\rho+\eta})^k|a_n|}{1-\sum_{n=2}^{\infty} n(\frac{\rho+(\mu+\lambda)(n-1)+\eta}{\rho+\eta})^k|a_n|},$$

which yields

$$(1-\alpha)-(1+\beta)|\frac{D_\lambda^k(\rho,\eta,\mu)f(z)}{z(D_\lambda^k(\rho,\eta,\mu)f(z))'}-1|$$

$$> \frac{(1-\alpha)-\sum_{n=2}^{\infty}[2n-n(\alpha-\beta)-(\beta+1)](\frac{\rho+(\mu+\lambda)(n-1)+\eta}{\rho+\eta})^k|a_n|}{1-\sum_{n=2}^{\infty} n(\frac{\rho+(\mu+\lambda)(n-1)+\eta}{\rho+\eta})^k|a_n|} \geq 0.$$

Hence the proof is complete.

# 3 Partial Sums

**Theorem 3.1** *If $f$ of the form (1) satisfies the condition (6) and $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, then*

$$\Re\{\frac{f(z)}{f_m(z)}\} \geq 1 - \frac{1}{C_{m+1}}, \tag{7}$$

*and*

$$C_n \geq \left\{ \begin{array}{cc} 1 & n = 2,3,...,m \\ C_{m+1} & n = m+1, m+2, ... \end{array} \right\},$$

*where*

$$C_n = \frac{[2n - n(\alpha - \beta) - (\beta + 1)]}{1 - \alpha}\left(\frac{\rho + (\mu + \lambda)(n - 1) + \eta}{\rho + \eta}\right)^k. \tag{8}$$

*The result in (7) is sharp for every m, with the extremal function*

$$f(z) = z + \frac{z^{m+1}}{C_{m+1}}. \tag{9}$$

**Proof:** We may write

$$\frac{1 + w(z)}{1 - w(z)} = C_{m+1}\left\{\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{C_{m+1}}\right)\right\}$$

$$= \left\{\frac{1 + \sum_{n=2}^{m} a_n z^{n-1} + C_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{m} a_n z^{n-1}}\right..$$

Then

$$w(z) = \frac{C_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}}{2 + 2\sum_{n=2}^{m} a_n z^{n-1} + C_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}},$$

and

$$|w(z)| \le \frac{C_{m+1}\sum_{n=m+1}^{\infty} |a_n|}{2 - 2\sum_{n=2}^{m} |a_n| + C_{m+1}\sum_{n=m+1}^{\infty} |a_n|}.$$

Now $|w(z)| \le 1$ if

$$2C_{m+1}\sum_{n=m+1}^{\infty} |a_n| \le 2 - 2\sum_{n=2}^{m} |a_n|,$$

which is equivalent to

$$\sum_{n=2}^{m} |a_n| + C_{m+1}\sum_{n=m+1}^{\infty} |a_n| \le 1. \tag{10}$$

It is suffices to show that the left hand side of (10) is bounded above by $\sum_{n=2}^{\infty} C_n|a_n|$, which is equivalent to

$$\sum_{n=2}^{m}(C_n - 1)|a_n| + \sum_{n=m+1}^{\infty}(C_n - C_{m+1})|a_n| \ge 0.$$

To see that the function $f$ given by (9) gives the sharp result, we observe for $z = re^{i\pi/n}$ that

$$\frac{f(z)}{f_m(z)} = 1 + \frac{z^m}{C_{m+1}}.$$

Letting $z \to 1^-$, we have

$$\frac{f(z)}{f_m(z)} = 1 - \frac{1}{C_{m+1}}.$$

Hence the proof is complete.

**Theorem 3.2** *If $f$ of the form (1) satisfies the condition (6) and $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, then*

$$\Re\{\frac{f_m(z)}{f(z)}\} \geq \frac{C_{m+1}}{1 + C_{m+1}}.$$

*The result is sharp for every $m$, with the extremal function $f(z)$given by (9).*

**Proof:** We may write

$$\frac{1 + w(z)}{1 - w(z)} = (1 + C_{m+1})\{\frac{f_m(z)}{f(z)} - \frac{C_{m+1}}{1 + C_{m+1}})\}$$

$$= \{\frac{1 + \sum_{n=2}^{m} a_n z^{n-1} - C_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}},$$

where

$$w(z) = \frac{(1 + C_{m+1}) \sum_{n=m+1}^{\infty} a_n z^{n-1}}{-(2 + 2\sum_{n=2}^{m} a_n z^{n-1} - (1 - C_{m+1}) \sum_{n=m+1}^{\infty} a_n z^{n-1})},$$

and

$$|w(z)| \leq \frac{(1 + C_{m+1}) \sum_{n=m+1}^{\infty} |a_n|}{2 - 2\sum_{n=2}^{m} |a_n| - (1 - C_{m+1}) \sum_{n=m+1}^{\infty} |a_n|}.$$

Now $|w(z)| \leq 1$ if

$$2C_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq 2 - 2\sum_{n=2}^{m} |a_n|,$$

which is equivalent to

$$\sum_{n=2}^{m} |a_n| + C_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq 1. \tag{11}$$

It is suffices to show that the left hand side of (11) is bounded above by $\sum_{n=2}^{\infty} C_n |a_n|$, which is equivalent to

$$\sum_{n=2}^{m} (C_n - 1)|a_n| + \sum_{n=m+1}^{\infty} (C_n - C_{m+1})|a_n| \geq 0.$$

Hence the proof is complete.

Putting $\eta = 1, \mu = 0$, in Theorem 3.1 and 3.2, respectively, we obtain the following corollary.

**Corollary 3.3** *If $f$ of the form (1) satisfies the condition (6) and $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, then*

$$\Re\{\frac{f(z)}{f_m(z)}\} \geq \frac{[2m - (m+1)(\alpha - \beta) - (\beta - 1)](\rho + \lambda + m + 1)^k - (\rho + 1)^k (1 - \alpha)}{[2m - (m+1)(\alpha - \beta) - (\beta - 1)](\rho + \lambda + m + 1)^k}.$$

*and*

$$\Re\{\frac{f_m(z)}{f(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](\rho+\lambda+m+1)^k}{(\rho+1)^k(1-\alpha)+[2m-(m+1)(\alpha-\beta)-(\beta-1)](\rho+\lambda+m+1)^k}.$$

*Putting $\rho = 1, \eta = 0$, and $\mu = 0$, in Theorem 3.1 and 3.2, respectively, we obtain the following corollary.*

**Corollary 3.4** *If $f$ of the form (1) satisfies the condition (6) and $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, then*

$$\Re\{\frac{f(z)}{f_m(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](1+\lambda m)^k-(1-\alpha)}{[2m-(m+1)(\alpha-\beta)-(\beta-1)](1+\lambda m)^k}.$$

*and*

$$\Re\{\frac{f_m(z)}{f(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](1+\lambda m)^k}{(1-\alpha)+[2m-(m+1)(\alpha-\beta)-(\beta-1)](1+\lambda m)^k}.$$

*Putting $\rho = 1, \eta = 0, \mu = 0$ and $\lambda = 1$, in Theorem 3.1 and 3.2, respectively, we obtain the following corollary.*

**Corollary 3.5** *If $f$ of the form (1) satisfies the condition (6) and $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, then*

$$\Re\{\frac{f(z)}{f_m(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+1)^k-(1-\alpha)}{[2m-(m+1)(\alpha-\beta)-(\beta-1)](1+\lambda m)^k}.$$

*and*

$$\Re\{\frac{f_m(z)}{f(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](1+\lambda m)^k}{(1-\alpha)+[2m-(m+1)(\alpha-\beta)-(\beta-1)](1+\lambda m)^k}.$$

*Putting $\rho = 1, \eta = 1, \lambda = 1$ and $\mu = 0$, in Theorem 3.1 and 3.2, respectively, we obtain the following corollary.*

**Corollary 3.6** *If $f$ of the form (1) satisfies the condition (6) and $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, then*

$$\Re\{\frac{f(z)}{f_m(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+2)^k-(2)^k(1-\alpha)}{[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+2)^k}.$$

*and*

$$\Re\{\frac{f_m(z)}{f(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+2)^k}{(2)^k(1-\alpha)+[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+2)^k}.$$

*Putting $\eta = 1, \lambda = 1$ and $\mu = 0$, in Theorem 3.1 and 3.2, respectively, we obtain the following corollary.*

**Corollary 3.7** *If $f$ of the form (1) satisfies the condition (6) and $\frac{f(z)}{z} \neq 0 (0<|z|<1)$, then*

$$\Re\{\frac{f(z)}{f_m(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+1+\rho)^k-(\rho+1)^k(1-\alpha)}{[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+1+\rho)^k}.$$

*and*

$$\Re\{\frac{f_m(z)}{f(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+1+\rho)^k}{(\rho+1)^k(1-\alpha)+[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+1+\rho)^k}.$$

**Theorem 3.8** *If $f$ of the form (1) satisfies the condition (6) and $\frac{f(z)}{z} \neq 0 (0<|z|<1)$, then*

$$(a)\Re\{\frac{f'(z)}{f'_m(z)}\} \geq 1-\frac{m+1}{C_{m+1}}, \tag{12}$$

$$(b)\Re\{\frac{f'_m(z)}{f'(z)}\} \geq \frac{C_{m+1}}{1+m+C_{m+1}},$$

*where*

$$C_n \geq \left\{ \begin{array}{cc} 1 & n=1,2,3,...,m \\ n\frac{C_{m+1}}{m+1} & n=m+1,m+2,... \end{array} \right\},$$

*and $C_n$ is defined by (8). The estimates in (12) and (13) are sharp with the extremal function given by (9).*

**Proof:** We prove only (a), which is similar to the proof of Theorem 3.1. The proof of (b) follows the pattern of that in Theorem 3.2. We write

$$\frac{1+w(z)}{1-w(z)} = C_{m+1}\{\frac{f'(z)}{f'_m(z)}-(1-\frac{1+m}{C_{m+1}})\}$$

$$=\{\frac{1+\sum_{n=2}^{m} na_n z^{n-1}+\frac{C_{m+1}}{m+1}\sum_{n=m+1}^{\infty} na_n z^{n-1}}{1+\sum_{n=2}^{m} na_n z^{n-1}},$$

where

$$w(z)=\frac{\frac{C_{m+1}}{m+1}\sum_{n=m+1}^{\infty} na_n z^{n-1}}{2+2\sum_{n=2}^{m} na_n z^{n-1}+\frac{C_{m+1}}{m+1}\sum_{n=m+1}^{\infty} na_n z^{n-1}},$$

and

$$|w(z)| \leq \frac{\frac{C_{m+1}}{m+1}\sum_{n=m+1}^{\infty} n|a_n|}{2-2\sum_{n=2}^{m} n|a_n|-\frac{C_{m+1}}{m+1}\sum_{n=m+1}^{\infty} n|a_n|}.$$

Now $|w(z)| \leq 1$ if and only if

$$\sum_{n=2}^{\infty} n|a_n|+\frac{C_{m+1}}{m+1}\sum_{n=m+1}^{\infty} n|a_n| \leq 1, \tag{14}$$

since the left hand side of (14) is bounded above by $\sum_{n=2}^{\infty} C_n|a_n|$. Hence the proof is complete.

Putting $\eta=1, \mu=0$, in Theorem 3.3, we obtain the following corollary.

**Corollary 3.9** *If $f$ of the form (1) satisfies the condition (6) and $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, then*

$$\Re\{\frac{f(z)}{f_m(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](\rho+\lambda+m+1)^k-(m+1)(\rho+1)^k(1-\alpha)}{[2m-(m+1)(\alpha-\beta)-(\beta-1)](\rho+\lambda+m+1)^k}.$$

*and*

$$\Re\{\frac{f_m(z)}{f(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](\rho+\lambda+m+1)^k}{(m+1)(\rho+1)^k(1-\alpha)+[2m-(m+1)(\alpha-\beta)-(\beta-1)](\rho+\lambda+m+1)^k}.$$

*Putting $\rho = 1, \eta = 0$, and $\mu = 0$, in Theorem 3.3, we obtain the following corollary.*

**Corollary 3.10** *If $f$ of the form (1) satisfies the condition (6) and $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, then*

$$\Re\{\frac{f(z)}{f_m(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](1+\lambda m)^k-(m+1)(1-\alpha)}{[2m-(m+1)(\alpha-\beta)-(\beta-1)](1+\lambda m)^k}.$$

*and*

$$\Re\{\frac{f_m(z)}{f(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](1+\lambda m)^k}{(m+1)(1-\alpha)+[2m-(m+1)(\alpha-\beta)-(\beta-1)](1+\lambda m)^k}.$$

*Putting $\rho = 1, \eta = 0, \mu = 0$ and $\lambda = 1$, in Theorem 3.3, we obtain the following corollary.*

**Corollary 3.11** *If $f$ of the form (1) satisfies the condition (6) and $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, then*

$$\Re\{\frac{f(z)}{f_m(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+1)^k-(m+1)(1-\alpha)}{[2m-(m+1)(\alpha-\beta)-(\beta-1)](1+\lambda m)^k}.$$

*and*

$$\Re\{\frac{f_m(z)}{f(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](1+\lambda m)^k}{(m+1)(1-\alpha)+[2m-(m+1)(\alpha-\beta)-(\beta-1)](1+\lambda m)^k}.$$

*Putting $\rho = 1, \eta = 1, \lambda = 1$ and $\mu = 0$, in Theorem 3.3, we obtain the following corollary.*

**Corollary 3.12** *If $f$ of the form (1) satisfies the condition (6) and $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, then*

$$\Re\{\frac{f(z)}{f_m(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+2)^k-(m+1)(2)^k(1-\alpha)}{[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+2)^k}.$$

*and*

$$\Re\{\frac{f_m(z)}{f(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+2)^k}{(m+1)(2)^k(1-\alpha)+[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+2)^k}.$$

*Putting $\eta = 1, \lambda = 1$ and $\mu = 0$, in Theorem 3.3, we obtain the following corollary.*

**Corollary 3.13** *If $f$ of the form (1) satisfies the condition (6) and $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, then*

$$\Re\{\frac{f(z)}{f_m(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+1+\rho)^k-(m+1)(\rho+1)^k(1-\alpha)}{[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+1+\rho)^k}.$$

*and*

$$\Re\{\frac{f_m(z)}{f(z)}\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+1+\rho)^k}{(m+1)(\rho+1)^k(1-\alpha)+[2m-(m+1)(\alpha-\beta)-(\beta-1)](m+1+\rho)^k}.$$

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