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Fixed Point Theorems for Weakly B-Contractive Mappings in Ordered Metric Spaces

V.S. Bright[1] and M. Marudai[2]

[1]Department of Mathematics, The American College
Madurai, Tamilnadu, India
E-mail: bright681957@gmail.com
[2]Department of Mathematics, Bharathidasan University
Trichy, Tamilnadu, India
E-mail: mmarudai@yahoo.co.in

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Abstract

The purpose of this paper is to present some fixed point results for weakly B-contractive mappings in a complete metric space endowed with a partial order.
Keywords: *Fixed point, Ordered metric space, weak B-contraction.*

1 Introduction

The Banach contraction mapping is one of the landmark results of functional analysis. It is widely known as the source of metric fixed point theory. Further its importance lies in its vast applicability in various branches of mathematics. Generalization of the above principle has been extensively investigated branch of research. In particular, V.S. Bright in [1] introduce the following definition.

Definition 1.1. *A mapping $T : X \to X$, where (X, d) is a metric space, is said to be a B-contraction if there exist positive real numbers α, β, γ such that $0 \leq \alpha + 2\beta + 2\gamma < 1$ for all $x, y \in X$. The following inequality holds:*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)] \tag{1.1}$$

Herein [1], it has been proved that if X is complete, then every B-contraction has a unique fixed point. On establishing this result there is no requirement of continuity of the B-contraction. Also in [1] V. S. Bright introduced a generalization of B-contraction given by the following definition.

Definition 1.2. *A mapping $T : X \to X$, where (X, d) is a metric space, is said to be weakly B-contractive or a* **weak B-contraction** *if for all $x, y \in X$ such that*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)] \\ - \psi[d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)] \qquad (1.2)$$

where $\psi : [0, \infty)^5 \to [0, \infty)$ is a continuous mapping such that $0 < \alpha+2\beta+2\gamma \leq 1$ and α, β and γ are non-zero positive numbers,
$\psi(x, y, z, u, v) = 0$ iff $x = y = z = u = v = 0$.

If we take $\psi(x, y, z, u, v) = \alpha_1 x+\beta_1(y+z)+\gamma_1(u+v)$, where $0 < \alpha_1+2\beta_1+2\gamma_1 < 1$ with $\alpha > \alpha_1$, $\beta > \beta_1$ and $\gamma > \gamma_1$ and α_1, β_1 and γ_1 are positive non-zero real numbers, then (1.2) reduces to (1.1). That is, weak B-contraction is a generalization of B-contraction.

In [1] V.S.Bright also proved that if X is complete, then every weak B-contraction has a unique fixed point. The purpose of this paper is to present the result in the context of ordered metric spaces.

2 Fixed Point Results: Nondecreasing Case

We start with the following definition.

Definition 2.1. *If $(X, \leq)$ is a partially ordered set $T : X \to X$ we say that T is monotonic nondecreasing if, for $x, y \in X$,*

$$x \leq y \Rightarrow Tx \leq Ty.$$

This definition coincides with the notion of nondecreasing function in the case where $X = R$ and $' \leq '$ represents the usual total order in R. Now let us present the following theorem.

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric. Let $T : X \to X$ be a continuous and nondecreasing such that*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)] \\ - \psi[d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)] \qquad \text{for } x \geq y \qquad (1.3)$$

where $\psi : [0,\infty)^5 \to [0,\infty)$ *is a continuous mapping such that* $0 < \alpha + 2\beta + 2\gamma \leq 1$ *and* α, β, γ *are nonzero positive real numbers,* $\psi(x,y,z,u,v) = 0$ *iff* $x = y = z = u = v = 0$. *If there exist* $x_0 \in X$ *with* $x_0 \leq Tx_0$, *then* T *has a fixed point.*

Proof: If $Tx_0 = x_0$, then the proof is finished. Suppose $x_0 < Tx_0$. Since $x_0 < Tx_0$ and T is a nondecreasing mapping we obtain by induction that $x_0 < Tx_0 \leq T^2x_0 \leq T^3x_0 \leq ... \leq T^nx_0 \leq T^{n+1}x_0 \leq ...$ put $x_{n+1} = Tx_n$. Then, for each integer $n \geq 1$, from (1.3) and, as elements x_{n-1} and x_n are comparable, we get

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1}) \\ &\leq \alpha d(x_n,x_{n-1}) + \beta[d(x_n,Tx_n) + d(x_{n-1},Tx_{n-1})] \\ &+ \gamma[d(x_n,Tx_{n-1}) + d(x_{n-1},Tx_n)] \\ &- \psi[d(x_n,x_{n-1}), d(x_n,Tx_n), d(x_{n-1},Tx_{n-1}), d(x_n,Tx_{n-1}), \\ & d(x_{n-1},Tx_n)] \\ &= \alpha d(x_n,x_{n-1}) + \beta[d(x_n,x_{n+1}) + d(x_{n-1},x_n)] + \gamma[d(x_n,x_n) \\ &+ d(x_{n-1},x_{n+1})] \\ &- \psi[d(x_n,x_{n-1}), d(x_n,x_{n+1}), d(x_{n-1},x_n), d(x_n,x_n), d(x_{n-1},x_{n+1})] \\ &= \alpha d(x_n,x_{n-1}) + \beta[d(x_n,x_{n+1}) + d(x_{n-1},x_n)] + \gamma[d(x_{n-1},x_{n+1})] \\ &- \psi[d(x_n,x_{n-1}), d(x_n,x_{n+1}), d(x_{n-1},x_n), 0, d(x_{n-1},x_{n+1})] \quad (1.4) \\ &\leq \alpha d(x_n,x_{n-1}) + \beta[d(x_n,x_{n+1}) + d(x_{n-1},x_n)] + \gamma d(x_{n-1},x_{n+1}) \\ &\leq \alpha d(x_n,x_{n-1}) + \beta[d(x_n,x_{n+1}) + d(x_{n-1},x_n)] + \gamma[(d(x_{n-1},x_n) \\ &+ d(x_n,x_{n+1})] \\ &\leq (\alpha+\beta+\gamma)d(x_n,x_{n-1}) + (\beta+\gamma)d(x_n,x_{n+1}) \end{aligned}$$

$$\begin{aligned} \{1-(\beta+\gamma)\}d(x_n,x_{n+1}) &\leq (\alpha+\beta+\gamma)d(x_n,x_{n-1}) \\ d(x_n,x_{n+1}) &\leq \frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)}d(x_n,x_{n-1}) \end{aligned}$$

$$\begin{aligned} &\alpha + 2\beta + 2\gamma \leq 1 \\ &\Rightarrow \alpha+\beta+\gamma \leq 1-(\beta+\gamma) \\ &\Rightarrow \frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} \leq 1. \end{aligned}$$

Therefore, $d(x_n,x_{n+1}) \leq d(x_n,x_{n-1})$.

Thus $\{d(x_{n+1},x_n)\}$ is a decreasing sequence of nonnegative real numbers and hence it is convergent.

Let

$$\lim_{n\to\infty} d(x_{n+1},x_n) = r \qquad (1.5)$$

Letting $n \to \infty$ in (1.4) we have

$$\begin{aligned} r &\leq \alpha r + \beta(r + r) + \gamma \lim_{n\to\infty} d(x_{n-1}, x_{n+1}) \\ \{1 - (\alpha + 2\beta)\}r &\leq \gamma \lim_{n\to\infty} d(x_{n-1}, x_{n+1}) \\ &\leq \gamma \lim_{n\to\infty} d(x_{n-1}, x_n) + \gamma \lim_{n\to\infty} d(x_n, x_{n+1}) \\ &= \gamma r + \gamma r = 2\gamma r \\ \therefore \{1 - (\alpha + 2\beta)\}r &\leq \gamma \lim_{n\to\infty} d(x_{n-1}, x_{n+1}) \\ &\leq 2\gamma r \end{aligned}$$

But $\alpha + 2\beta + 2\gamma \leq 1 \Rightarrow 2\gamma \leq 1 - (\alpha + 2\beta)$. Therefore

$$2\gamma r \leq \gamma \lim_{n\to\infty} d(x_{n-1}, x_{n+1}) \leq 2\gamma r.$$

That is, $2r \leq \lim_{n\to\infty} d(x_{n-1}, x_{n+1}) \leq 2r$.

$$\Rightarrow \lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 2r \qquad (1.6)$$

Again, making $n \to \infty$ in (1.4) and using (1.5), (1.6) and the continuity of ψ we obtain

$$\begin{aligned} r &\leq \alpha r + 2\beta r + 2\gamma r - \psi(r, r, r, 0, 2r) \\ \{1 - (\alpha + 2\beta + 2\gamma)\}r &\leq -\psi(r, r, r, 0, 2r) \leq 0 \\ \alpha + 2\beta + 2\gamma &\leq 1 \\ &\Rightarrow 0 \leq 1 - (\alpha + 2\beta + 2\gamma) \end{aligned}$$

Hence

$$\begin{aligned} 0 &\leq -\psi(r, r, r, 0, 2r) \\ &\leq 0 \\ \psi(r, r, r, 0, 2r) &= 0 \end{aligned}$$

By the definition of ψ, $r = 0$. Thus we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \qquad (1.7)$$

In what follows, we will prove that $\{x_n\}$ is a cauchy sequence. If otherwise, then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that for any k

$$d(x_{n(k)}, x_{m(k)}) \geq \epsilon. \qquad (1.8)$$

Further, corresponding to $m(k)$ we can choose $n(k)$ in such a way that it is a smallest integer with $n(k) > m(k)$ and satisfying (1.8). Then

$$d(x_{n(k)-1}, x_{m(k)}) < \epsilon. \qquad (1.9)$$

Using $(1.8), (1.9)$ and triangle inequality, we have

$$\begin{aligned}\epsilon &\le d(x_{n(k)}, x_{m(k)})\\ &\le d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)})\\ &< d(x_{n(k)}, x_{n(k)-1}) + \epsilon.\end{aligned}$$

Making $k \to \infty$ the above inequality and using (1.7) we get

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \epsilon. \qquad (1.10)$$

Again, the triangular inequality gives us

$$d(x_{m(k)}, x_{n(k)-1}) \le d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1}).$$

and

$$d(x_{m(k)-1}, x_{n(k)}) \le d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}).$$

letting $k \to \infty$ in the above two inequalities and using (1.7) and (1.10) we get

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon. \qquad (1.11)$$

As $n(k) > m(k)$ and $x_{n(k)-1}$ and $x_{m(k)-1}$ are comparable using (1.3) we have

$$\begin{aligned}\epsilon &\le d(x_{n(k)}, x_{m(k)})\\ &= d(Tx_{n(k)-1}, Tx_{m(k)-1})\\ &\le \alpha d(x_{n(k)-1}, x_{m(k)-1}) + \beta[d(x_{n(k)-1}, Tx_{n(k)-1}) + d(x_{m(k)-1}, Tx_{m(k)-1})]\\ &+ \gamma[d(x_{n(k)-1}, Tx_{m(k)-1}) + d(x_{m(k)-1}, Tx_{n(k)-1})]\\ &- \psi[d(x_{n(k)-1}, x_{m(k)-1}), d(x_{n(k)-1}, x_{n(k)}), d(x_{m(k)-1}, x_{m(k)}),\\ &d(x_{n(k)-1}, x_{m(k)}), d(x_{m(k)-1}, x_{n(k)})]\\ &= \alpha d(x_{n(k)-1}, x_{m(k)-1}) + \beta[d(x_{n(k)-1}, x_{n(k)}) + d(x_{m(k)-1}, x_{m(k)})]\\ &+ \gamma[d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)-1}, x_{n(k)})]\\ &- \psi[d(x_{n(k)-1}, x_{m(k)-1}), d(x_{n(k)-1}, x_{n(k)}), d(x_{m(k)-1}, x_{m(k)}),\\ &d(x_{n(k)-1}, x_{m(k)}), d(x_{m(k)-1}, x_{n(k)})]\end{aligned}$$

Making $k \to \infty$ taking into account $(1.10), (1.11)$ and continuity of ψ we have

$$\epsilon \le \alpha \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) + 2\gamma\epsilon - \psi[\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}), 0, 0, \epsilon, \epsilon]$$

$$\begin{aligned}\epsilon &\le \alpha \lim_{k\to\infty} [d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1})] + 2\gamma\epsilon\\ &- \psi[\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}), 0, 0, \epsilon, \epsilon]\\ &\le \alpha\epsilon + 2\gamma\epsilon - \psi[\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}), 0, 0, \epsilon, \epsilon]\end{aligned}$$

Now

$$\begin{aligned} d(x_{m(k)}, x_{n(k)-1}) &\le d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)-1}) \\ \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) &\le \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) \\ &\le \lim_{k\to\infty} \{d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)-1})\} \\ &= \lim_{k\to\infty} \{d(x_{m(k)}, x_{n(k)-1})\} = \epsilon \quad (by(1.10)) \end{aligned}$$

Hence

$$\epsilon \le \lim_{n\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) \le \epsilon$$

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon$$

$$\begin{aligned} \epsilon &\le (\alpha + 2\gamma)\epsilon - \psi(\epsilon, 0, 0, \epsilon, \epsilon) \\ \epsilon &\le \epsilon - \psi(\epsilon, 0, 0, \epsilon, \epsilon) < \epsilon, \quad \textit{since} \quad \psi \ge 0 \\ \Rightarrow \psi(\epsilon, 0, 0, \epsilon, \epsilon) &= 0 \\ \Rightarrow \epsilon &= 0 \quad \text{a contradiction, since } \epsilon > 0 \end{aligned}$$

This proves that $\{x_n\}$ is a cauchy sequence. Since X is complete, there exist $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Moreover the continuity of T implies that $Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z$ and this proves that z is a fixed point for T. □

In what follows we prove that theorem (2.2), is still valid for T not necessarily continuous, assuming the following hypothesis in X (which appears in theorem 1 of [9]): If $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$ then $x_n \le x$ for all $n \in N$ (1.12).

Theorem 2.3. *Let $(X, \le)$ be a partially ordered set and suppose that there exist a metric d in X such that (X, d) is a complete metric space. Assume that X satisfies* (1.12)*. Let $T : X \to X$ be a nondecreasing mapping such that*

$$\begin{aligned} d(Tx, Ty) \le{}& \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)] \\ &- \psi[d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)] \qquad \textit{for } x \ge y \end{aligned}$$

where $\psi : [0, \infty)^5 \to [0, \infty)$ is a continuous mapping such that $0 < \alpha + 2\beta + 2\gamma \le 1$ and α, β, γ are nonzero positive real numbers, $\psi(x, y, z, u, v) = 0$ iff $x = y = z = u = v = 0$. If there exists $x_0 \in X$ with $x_0 \le Tx_0$, then T has a fixed point.

Proof: Following the proof of theorem 2.2, we only have to check $Tz = z$. As $\{x_n\}$ is a nondecreasing sequence in X and $\{x_n\} \to z$. Then the condition (1.12) gives that $x_n \le z$ for every $n \in N$. The contractive condition (1.3) gives us

$$\begin{aligned}
d(x_{n+1}, Tz) &= d(Tx_n, Tz) \\
&\le \alpha d(x_n, z) + \beta[d(x_n, Tx_n) + d(z, Tz)] + \gamma[d(x_n, Tz) + d(z, Tx_n)] \\
&\quad - \psi[d(x_n, z), d(x_n, Tx_n), d(z, Tz), d(x_n, Tz), d(z, Tx_n)] \\
d(x_{n+1}, Tz) &\le \alpha d(x_n, z) + \beta[d(x_n, x_{n+1}) + d(z, Tz)] + \gamma[d(x_n, Tz) + d(z, x_{n+1})] \\
&\quad - \psi[d(x_n, z), d(x_n, x_{n+1}), d(z, Tz), d(x_n, Tz), d(z, x_{n+1})]
\end{aligned}$$

Letting $n \to \infty$ and using the continuity of ψ and d, we have

$$\begin{aligned}
d(z, Tz) &\le (\beta + \gamma)d(z, Tz) - \psi[d(z, z), d(z, z), \\
&\quad d(z, Tz), d(z, Tz), d(z, z)] \\
&= (\beta + \gamma)d(z, Tz) - \psi[0, 0, d(z, Tz), d(z, Tz), 0] \\
d(z, Tz) &\le d(z, Tz) - \psi[0, 0, d(z, Tz), d(z, Tz), 0] \le d(z, Tz)
\end{aligned}$$

$$\begin{aligned}
\psi[0, 0, d(z, Tz), d(z, Tz), 0] &= 0 \\
\Rightarrow d(z, Tz) &= 0 \\
\Rightarrow Tz &= z
\end{aligned}$$

This completes the proof. □

Note 2.4. *The above two theorems do not guarantee uniqueness of the fixed point. The following example will illustrate it.*

Let $X = \{(1, 0), (0, 1), (-2, 2), (2, -2)\} \subset R^2$ *and consider the usual order* $(x, y) \le (z, t)$ *iff* $x \le z$ *and* $y \le t$. *Thus* $(X, \le)$ *is a partially ordered set whose different elements are not comparable. Besides,* (X, d_2) *is a complete metric space considering* d_2 *the euclidean distance. The identity map* $T(x, y) = (x, y)$ *is trivially continuous and nondecreasing and condition* (1.3) *of theorem* 2.2 *is satisfied, since elements in* X *are only comparable to themselves. Moreover* $(1, 0) \le T(1, 0) = (1, 0)$ *and* T *has four fixed points.*

Note 2.5. *In order to have a unique fixed point for the above two theorems (2.2) and (2.3) we add the condition (it appears in [10]) for* $x, y \in X$ *there exists a lower bound or upper bound.*

In [9] it is proved that the above mentioned condition is equivalent to for $x, y \in X$ *there exists* $z \in x$ *which is comparable to* x *and* y *(1.13).*

Theorem 2.6. *Adding condition* (1.13) *to the hypothesis of theorem* 2.2*(or theorem* 2.3*). We obtain the uniqueness of the fixed point of* T.

Proof: Suppose that there exist $z, y \in X$ which are fixed points of T. We distinguish two cases:

Case 1: If y is comparable to z, then $T^n y = y$ is comparable to $T^n z = z$ for $n = 1, 2, 3, ...$ and

$$\begin{aligned}
d(y,z) &= d(T^n y, T^n z) \quad , y \geq z \\
&\leq \alpha d(T^{n-1}y, T^{n-1}z) + \beta[d(T^{n-1}y, T^n y) + d(T^{n-1}z, T^n z)] \\
&+ \gamma[d(T^{n-1}y, T^n z) + d(T^{n-1}z, T^n y)] \\
&- \psi[d(T^{n-1}y, T^{n-1}z), d(T^{n-1}y, T^n y), d(T^{n-1}z, T^n z), d(T^{n-1}y, T^n z), \\
&d(T^{n-1}z, T^n y)] \\
&= \alpha d(y,z) + \beta[y,y) + d(z,z)] + \gamma[d(y,z) + d(z,y)] \\
&- \psi[d(y,z), d(y,y), d(z,z), d(y,z), d(z,y)] \\
&= (\alpha + 2\gamma) d(y,z) - \psi[d(y,z), 0, 0, d(y,z), d(z,y)] \\
d(y,z) &\leq d(y,z) - \psi[d(y,z), 0, 0, d(y,z), d(z,y)] \leq d(y,z),
\end{aligned}$$

for $\psi \geq 0$ and $\alpha + 2\beta + 2\gamma \leq 1$ and α, β, γ are non zero positive real numbers.

$$\begin{aligned}
\Rightarrow \psi[d(y,z), 0, 0, d(y,z), d(z,y)] = 0 \\
\Rightarrow d(y,z) = 0 \\
\Rightarrow y = z
\end{aligned}$$

Case 2: If y is not comparable to z then (by (1.13)) there exist $x \in X$ comparable to y and z. Monotonicity of T implies that $T^n x$ is comparable to $T^n y = y$ and $T^n z = z$ for $n = 1, 2, 3, ...$

$$\begin{aligned}
d(z, T^n x) &= d(T^n z, T^n x) \\
&\leq \alpha d(T^{n-1}z, T^{n-1}x) + \beta[d(T^{n-1}z, T^n z) + d(T^{n-1}x, T^n x)] \\
&+ \gamma[d(T^{n-1}z, T^n x) + d(T^{n-1}x, T^n z)] \\
&- \psi[d(T^{n-1}z, T^{n-1}x), d(T^{n-1}z, T^n z), d(T^{n-1}x, T^n x), d(T^{n-1}z, T^n x), \\
&d(T^{n-1}x, T^n z)] \\
&= \alpha d(z, T^{n-1}x) + \beta[d(z,z) + d(T^{n-1}x, T^n x)] + \gamma[d(z, T^n x) \\
&+ d(T^{n-1}x, z)] \\
&- \psi[d(z, T^{n-1}x), d(z,z), d(T^{n-1}x, T^n x), d(z, T^n x), d(T^{n-1}x, z)] \\
&= \alpha d(z, T^{n-1}x) + \beta d(T^{n-1}x, T^n x) + \gamma[d(z, T^n x) + d(T^{n-1}x, z)] \\
&- \psi[d(z, T^{n-1}x), 0, d(T^{n-1}x, T^n x), d(z, T^n x), d(T^{n-1}x, z)] \\
&\leq \alpha d(z, T^{n-1}x) + \beta[d(T^{n-1}x, z) + d(z, T^n x)] \\
&+ \gamma[d(z, T^n x) + d(T^{n-1}x, z)] \\
&- \psi[d(z, T^{n-1}x), 0, d(T^{n-1}x, T^n x), d(z, T^n x), d(T^{n-1}x, z)] \qquad (1.14)
\end{aligned}$$

$$\begin{aligned}
\therefore \; d(z, T^n x) &\leq \alpha d(z, T^{n-1}x) + \beta[d(T^{n-1}x, z) + d(z, T^n x)] \\
&+ \gamma[d(z, T^n x) + d(T^{n-1}x, z)]
\end{aligned}$$

$$\{1-(\beta+\gamma)\}d(z,T^nx) \leq (\alpha+\beta+\gamma)d(z,T^{n-1}x)$$
$$d(z,T^nx) \leq \frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)}d(z,T^{n-1}x)$$
$$\alpha+\beta+\gamma \leq 1$$
$$\Rightarrow \alpha+\beta+\gamma \leq 1-(\beta+\gamma)$$
$$\frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} \leq 1$$

Therefore $d(z,T^nx) \leq d(z,T^{n-1}x)$. This proves that the nonnegative decreasing sequence $\{d(z,T^nx)\}$ is convergent. Put $\lim_{n\to\infty} d(z,T^nx) = r$. Hence we let $n \to \infty$ in (1.14) and taking into account the continuity of ψ we obtain

$$\begin{aligned} r &\leq \alpha r + \beta(r+r) + \gamma(r+r) - \psi(r,0,\lim_{n\to\infty} d(T^{n-1}x,T^nx),r,r) \\ &= (\alpha+2\beta+2\gamma)r - \psi(r,0,\lim_{n\to\infty} d(T^{n-1}x,T^nx),r,r) \\ &\leq r - \psi(r,0,\lim_{n\to\infty} d(T^{n-1}x,T^nx),r,r) \leq r \\ \Rightarrow r &\leq r - \psi(r,0,\lim_{n\to\infty} d(T^{n-1}x,T^nx),r,r) \leq r \end{aligned}$$

$\Rightarrow \psi(r,0,\lim_{n\to\infty} d(T^{n-1}x,T^nx),r,r) = 0 \Rightarrow r = 0$ and $\lim_{n\to\infty} d(T^{n-1}x,T^nx) = 0$. Consequently, $\lim_{n\to\infty} d(z,T^nx) = 0$. Analogously, it can be proved that $\lim_{n\to\infty} d(y,T^nx) = 0$. Since $\lim_{n\to\infty} d(z,T^nx) = 0 \Rightarrow \lim_{n\to\infty} T^nx = z$ *and* $\lim_{n\to\infty} d(y,T^nx) = 0 \Rightarrow \lim_{n\to\infty} T^nx = y$. By uniqueness of limit gives us $y = z$. This finishes the proof. □

Remark 2.7. *Notice that if $(X,\leq)$ is a totally ordered set, then the condition (1.13) is obviously satisfied then we obtain the uniqueness of the fixed point.*

Remark 2.8. *From the above theorem 2.2 or 2.3 or 2.6 as $\psi : [0,\infty)^5 \to [0,\infty)$ defined by $\psi(x,y,z,u,v) = \alpha_1 x + \beta_1(y+z) + \gamma_1(u+v)$ where $\alpha > \alpha_1$ and $\beta > \beta_1$ and $\gamma > \gamma_1$. Suppose that $\alpha_1 + 2\beta_1 + 2\gamma_1 \in (0,1)$ and α_1, β_1 and γ_1 are non zero positive real numbers. It follows that weak B-contraction changes into B-contraction mapping and the condition (1.3) of theorem* 2.2 *can be rewritten as*

$$\begin{aligned} d(Tx,Ty) \;\leq\; & \alpha d(x,y) + \beta[d(x,Tx)+d(y,Ty)] + \gamma[d(x,Ty)+d(y,Tx)] \\ & \textit{for } x \geq y \textit{ and } 0 < \alpha+2\beta+2\gamma < 1. \qquad (1.15) \end{aligned}$$

and theorem 2.2 or 2.3 or 2.6 will be true for this particular case (1.15), namely, B-contraction (1.1) in the context of ordered metric spaces.

3 Fixed Point Results: Nonincreasing Case

In this section we present a fixed point theorem for weakly B-contractive mappings when the operator T is nonincreasing. We begin with the following definition.

Definition 3.1. *If $(X, \leq)$ is a partially ordered set and $: X \to X$ we say that T is monotone nonincreasing if for $x, y \in X$, $x \leq y \Rightarrow Tx \geq Ty$.*

The main result of the section is the following theorem:

Theorem 3.2. *Let $(X, \leq)$ be a partially ordered set satisfying the condition (1.13) and suppose that there exist a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a nonincreasing mapping such that*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)] \\ - \psi[d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)] \qquad \text{for } x \geq y \qquad (1.16)$$

where $\psi : [0, \infty)^5 \to [0, \infty)$ is a continuous mapping such that $0 < \alpha + 2\beta + 2\gamma \leq 1$ and α, β, γ are nonzero positive real numbers, $\psi(x, y, z, u, v) = 0$ iff $x = y = z = u = v = 0$. If there exist $x_0 \in X$ with $x_0 \leq Tx_0$ or $x_0 \geq Tx_0$, then $inf\{d(x, Tx) : x \in X\} = 0$ if, in addition, X is compact and T is continuous, then T has a unique fixed point.

Proof: If $Tx_0 = x_0$ it is obvious that $inf\{d(x, Tx) : x \in X\} = 0$. Suppose that $x_0 < Tx_0$(the same argument serves for $x_0 < Tx_0$). By virtue of T being nonincreasing the consecutive terms of the sequence $T^n(x_0)$ are comparable using (1.16) we can obtain

$$d(T^{n+1}x_0, T^n x_0) \leq \alpha d(T^n x_0, T^{n-1}x_0) + \beta[d(T^n x_0, T^{n+1}x_0) + d(T^{n-1}x_0, T^n x_0)] \\ + \gamma[d(T^n x_0, T^n x_0) + d(T^{n-1}x_0, T^{n+1}x_0)] \\ - \psi[d(T^n x_0, T^{n-1}x_0), d(T^n x_0, T^{n+1}x_0), d(T^{n-1}x_0, T^n x_0), \\ d(T^n x_0, T^n x_0), d(T^{n-1}x_0, T^{n+1}x_0)]$$

$$= \alpha d(T^n x_0, T^{n-1}x_0) + \beta[d(T^n x_0, T^{n+1}x_0) + d(T^{n-1}x_0, T^n x_0)] \\ + \gamma[d(T^{n-1}x_0, T^{n+1}x_0)] \\ - \psi[d(T^n x_0, T^{n-1}x_0), d(T^n x_0, T^{n+1}x_0), d(T^{n-1}x_0, T^n x_0), 0, d(T^{n-1}x_0, T^{n+1}x_0)] \\ \leq \alpha d(T^n x_0, T^{n-1}x_0) + \beta[d(T^n x_0, T^{n+1}x_0) + d(T^{n-1}x_0, T^n x_0)] \\ + \gamma[d(T^{n-1}x_0, T^{n+1}x_0)]$$

Now

$$d(T^{n+1}x_0, T^n x_0) \leq \alpha d(T^n x_0, T^{n-1}x_0) + \beta[d(T^n x_0, T^{n+1}x_0) + d(T^{n-1}x_0, T^n x_0)] \\ + \gamma[d(T^{n-1}x_0, T^n x_0) + d(T^n x_0, T^{n+1}x_0)]$$

Hence

$$\{1-(\beta+\gamma)\}d(T^{n+1}x_0,T^nx_0) \leq (\alpha+\beta+\gamma)d(T^{n-1}x_0,T^nx_0)$$
$$d(T^{n+1}x_0,T^nx_0) \leq \frac{(\alpha+\beta+\gamma)}{\{1-(\beta+\gamma)\}}d(T^{n-1}x_0,T^nx_0)$$

$0<\alpha+2\beta+2\gamma \leq 1, \Rightarrow 0<\dfrac{(\alpha+\beta+\gamma)}{\{1-(\beta+\gamma)\}} \leq 1.$
Therefore $d(T^{n+1}x_0,T^nx_0) \leq d(T^nx_0,T^{n-1}x_0)$. From this inequality we have that $\{d(T^{n+1}x_0,T^nx_0)\}$ is a nonnegative decreasing sequence with limit $r \geq 0$. Using a similar argument that in theorem 2.2 we can prove that $r=0$. This means that $\lim_{n\to\infty} d(T^{n+1}x_0,T^nx_0)=0$ and, consequently $inf\{d(x,Tx):x\in X\}=0$. This finishes the first part of our theorem.

Now suppose X is compact and T is continuous taking into the account that the mapping

$$X \to R^+$$
$$x \to d(x,Tx)$$

is continuous (note that the mapping can be obtained as

$$X \to X\times X \to R^+$$
$$x \to (x,Tx) \to d(x,Tx),$$

and obviously,this composition of mapping is continuous because T is continuous) and since X is compact, we can find $z\in X$ such that

$$d(z,Tz)=inf\{d(x,Tx):x\in X\}.$$

Taking into account the first part of the theorem $d(z,Tz)=0$ and therefore z is a fixed point of T. The uniqueness of the fixed point is proved as in theorem 2.6. □

Remark 3.3. *An analogous result in the nonincreasing case cannot be obtained using a similar reasoning in Theorem 2.2 as the proof of Cauchy character of sequence $\{x_n\}$ fails since $x_{n(k)-1}$ and $\{x_{m(k)-1}\}$ cannot be comparable if T is a nonincreasing operator.*

4 Examples

In this section we present some examples which illustrate our results.

Example 4.1. *Let $X=\{(0,1),(1,0),(\frac{1}{2},\frac{1}{2})\}\subseteq R^2$ with the euclidean distance d_2. (X,d_2) is obviously, a complete metric space. Moreover, we consider* *the order $\leq$ in X given by $R = \{(x,x) : x \in X\}$. Notice that the elements in X are only comparable to themselves. Also we consider $T : X \to X$ is given by*

$$T(1,0) = (0,1)$$

$$T(0,1) = (1,0)$$

$$T\left(\frac{1}{2},\frac{1}{2}\right) = \left(\frac{1}{2},\frac{1}{2}\right).$$

Obviously T is a continuous nondecreasing mapping and moreover $(\frac{1}{2},\frac{1}{2}) \leq T(\frac{1}{2},\frac{1}{2})$. As the elements in X are only comparable to themselves, condition (1.3) is obviously satisfied. Finally, theorem 2.2 gives as the existence of the fixed point of T (which is obviously the point $\left(\frac{1}{2},\frac{1}{2}\right)$). On the other hand,

$$\begin{aligned} d_2(T(1,0),T(0,1)) &\leq \alpha d_2((1,0),(0,1)) + \beta[d_2((1,0),T(1,0)) + d_2((0,1),T(0,1))] \\ &+ \gamma[d_2((1,0),T(0,1)) + d_2((0,1),T(1,0))] \\ &- \psi[d_2((1,0),(0,1)), d_2((1,0),T(1,0)), d_2((0,0),T(0,1)), 0, 0] \\ &= \alpha\sqrt{2} + \beta(\sqrt{2}+\sqrt{2}) - \psi[\sqrt{2},\sqrt{2},\sqrt{2},0,0] \\ &= \sqrt{2}(\alpha + 2\beta) - \psi[\sqrt{2},\sqrt{2},\sqrt{2},0,0] \end{aligned}$$

But $d_2(T(1,0),T(0,1)) = \sqrt{2}$, $\sqrt{2} \leq \sqrt{2} - \psi[\sqrt{2},\sqrt{2},\sqrt{2},0,0]$.
But $\psi[\sqrt{2},\sqrt{2},\sqrt{2},0,0] > 0$. This implies the operator T is not a weak B-contraction. (See definition 1.2) consequently, this example cannot be treated by the main result of [1] because herein definition 1.2 can only apply those elements which are comparable..

Notice that in the example we obtain uniqueness of the fixed point and condition (1.13) appearing theorem 2.6 is not satisfied here $(X,\leq)$. This proves that condition (1.13) is not a necessary condition for the uniqueness of the fixed point.

Example 4.2. *Consider the space X with the euclidean distance d_2 and with order given by $R = \{(x,x) : x \in X\}$. Let $X = \{(0,1),(1,0),(1,1)\}$. Let T be the operator $T : X \to X$ defined by $T(0,1) = (0,1)$, $T(1,1) = (0,1)$ and $T(1,0) = (1,0)$. In what follows, we prove that T satisfies the condition (1.3) namely $x \geq y$ appearing in theorem 2.2. In fact, for $(0,1) \leq (1,1)$*

$$d_2(T(0,1),T(1,1)) = d_2((0,1),(0,1) = 0$$

and, consequently, condition (1.3) satisfied also $(0,1) \leq T(0,1)$, theorem 2.2 says as that T has a fixed point(in this case, $(0,1)$ and $(1,0)$ are the fixed points of T). Notice that, in this case, we have not uniqueness of the fixed point. Further $(X,\leq)$ does satisfy condition (1.13) of theorem 2.6. On the other *hand, the operator T is not a weak B-contraction, eventhough* $(1,0) \leq (1,1)$*. For,* $d_2(T(1,0),T(1,1)) = d_2((1,0),(0,1)) = \sqrt{2}$ *and*

$$\begin{aligned} d_2((1,0),(0,1)) = \sqrt{2} \\ &\leq \alpha d_2((1,0),(1,1)) + \beta[d_2((1,0),T(1,0)) + d_2((1,1),T(1,1))] \\ &+ \gamma[d_2((1,0),T(1,1)) + d((1,1),T(1,0))] \\ &- \psi[d_2((1,0),(1,1)), d_2((1,0),T(1,0)), d_2((1,1),T(0,1)), \\ &d_2((1,0),T(0,1)), d((1,1),T(1,0))] \\ i.e, \sqrt{2} &\leq \alpha + \beta + \gamma(\sqrt{2}+1) - \psi[1,0,1,\sqrt{2},1] \qquad (1.17) \\ i.e., \quad \sqrt{2} &\leq \alpha + \beta + \gamma + \sqrt{2}\gamma \\ \Rightarrow \sqrt{2} &\leq \frac{\alpha+\beta+\gamma}{1-\gamma} \end{aligned}$$

If $\dfrac{\alpha+\beta+\gamma}{1-\gamma} < 1 \Rightarrow \alpha+\beta+2\gamma < 1$ which is true, since α, β and γ are non zero positive real numbers(1.3).
Hence $\alpha+\beta+\gamma+\sqrt{2}\gamma < 1$ and $\psi \geq 0$, now by (1.16) $\Rightarrow\Leftarrow$.
Hence T is not a weak B-contraction.

Note 4.3. *Now we shall give some particular cases of B-contraction and weakly B-contractions.*
In B-contraction, if $\alpha = 0$ *and* $\beta = 0$*, then* $2\gamma < 1$ *implies* $\gamma < \frac{1}{2}$*.*
Therefore we get a C-contraction. That is

$$d(Tx,Ty) \leq \gamma\{d(x,Ty) + d(y,Tx)\}$$

for all $x, y \in X$ *and* $\gamma \in [0, \frac{1}{2})$*. The corresponding weak C-Contraction is*

$$d(Tx,Ty) \leq \gamma\{d(x,Ty) + d(y,Tx)\} - \psi\{d(x,Ty), d(y,Tx)\}$$

for all $x, y \in X$ *and* $\gamma \in (0, \frac{1}{2}]$ *and in particular*

$$d(Tx,Ty) \leq \frac{1}{2}\{d(x,Ty) + d(y,Tx)\} - \psi\{d(x,Ty), d(y,Tx)\}$$

where $\psi : [0,\infty)^2 \to [0,\infty)$ *is continuous such that* $\psi(x,y) = 0$ *if and only if* $x = y = 0$*.*

Hence whatever theorem proved in this paper with regard to weak B-contraction is also true for the theorem on weak C-contraction in [8].
Further, many other contractions such as Kannan [6], weak S-contractions [9] etc can follow from weak B-contraction see [1] and [7].
Thus weak B-contraction is considered to be the generalization of all other so-called contractions.

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