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Periodic Solution for certain Nonlinear System of Volterra Integral Equations

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Abstract

In this paper, the numerical-analytic method has been used by (Samoilenko A. M.) to investigate the existence and approximation of periodic solutions for certain of nonlinear system of volterra integral equations. Also these methods could be developed and extended throughout the study. Thus, the nonlinear integral equation that we have introduced in this study become more general than those introduced by Butris R. N.

Keywords: *Periodic Solution, Integral equation, Numerical-analytic method.*

I Introduction

Integral equations are one of the most useful mathematical tools in both pure and applied analysis. This is particularly true of problems in mechanical vibrations and the related fields of engineering and mathematical physics, where they are not only useful but often indispensable even for numerical computations.

To avoid some of the difficulties indicated in the previous section, Vito Volterra investigated the solution of integral equations in which the kernel satisfies the condition.

$$K(t,x) \equiv 0 \quad , \text{if} \quad x > t.$$

This corresponds to the simple case of a system of algebraic linear equations where the elements of the determinant above the main diagonal are all zero. The integral equations

$$\phi(t) - \lambda \int_0^t K(t,x)\phi\,(x)dx = f(t)$$

and

$$\int_0^t K(t,x)\phi(x)dx = f(t)$$

are called integral equations of the second and first kind, respectively. Integral equations made great affection in developing integral differential equations that have a great role in mathematical analysis and functional analysis [2, 3].

Samoilenko [4], assumes the numerical analytic method to study the periodic solutions for ordinary differential equations and their algorithm structure. This method includes uniformly sequences of periodic functions and the result is the use of the periodic solutions on a wide range which is different from the processes in industry and technology.

Consider the following system of nonlinear volterra integral equation which has the form:

$$u(t,u_0) = f(t) + \int_0^t G(t,s)f(s,u(s,u_0), \int_{-\infty}^{s} g(s,\tau,u(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u(\tau,u_0))d\tau)\, ds \quad \cdots (1)$$

.

Suppose that the functions

$$f(t,u,v,w) = (f_1(t,u,v,w), f_2(t,u,v,w), \cdots, f_n(t,u,v,w)),$$

$$g(t,s,u) = (g_1(t,s,u), g_2(t,s,u), \cdots, g_n(t,s,u)), h(t,s,u) = (h_1(t,s,u), h_2(t,s,u), \cdots, h_n(t,s,u))$$

and $f(t) = (f_1(t), f_2(t), \cdots, f_n(t))$ are defined on the domain

$$(t,u,v,w) \in R^1 \times D \times D_1 \times D_2 = (-\infty,\infty) \times D \times D_1 \times D_2 \quad \cdots (2)$$

Which are continuous functions in t, u, v, w and periodic in t of period T.

i. e. $\{f(t+T,u,v,w)=f(t,u,v,w)\}$, also $a(t)$ and $b(t)$ are continuous and periodic in t of period T,where D_1 and D_2 are closed and bounded domains subsets of Euclidean space R^n.

Suppose that the functions $f(t,u,v,w), g(t,s,u)$ and $h(t,s,u)$ satisfies the following inequalities:

$$\left.\begin{aligned} &\|f(t,u,v,w)\| \le M\,, \\ &\|g(t,s,u)\| \le N_1\,, \\ &\|h(t,s,u)\| \le N_2\,. \end{aligned}\right\} \quad \cdots (3)$$

$$\|f(t,u_1,v_1,w_1)-f(t,u_1,v_1,w_1)\| \le L_1\|u_1-u_2\|+L_2\|v_1-v_2\|+L_3\|w_1-w_2\|\ ; \quad \dots (4)$$

$$\|g(t,s,u_1)-g(t,s,u_2)\| \le \|K(t,s)\|\|u_1-u_2\|\ ; \quad \cdots (5)$$

$$\|h(t,s,u_1)-h(t,s,u_2)\| \le \|H(t,s)\|\|u_1-u_2\|\ . \quad \cdots (6)$$

$\forall\ t \in R^1, u,u_1,u_2 \in D, v,v_1,v_2 \in D_1$

And $w,w_1,w_2 \in D_2$ where M , N_1,N_2,L_1 , L_2 and $L_3$are positive constants. Furthermore $G(t,s),K(t,s)$ and $H(t,s)$ are $(n\times n)$ positive matrices which are defined and continuous and periodic in t, s in the domain $(R^1\times R^1)$ and satisfy the following conditions:

$$\|G(t,s)\| \le R < \infty \quad \cdots (7)$$

$$\int_{-\infty}^{t}\|K(t,s)\|ds \le Q_1 < \infty \quad \cdots (8)$$

$$\int_{a(t)}^{b(t)}\|H(t,s)\|ds \le Q_2 < \infty \quad \cdots (9)$$

Where$-\infty < 0 \le s \le t \le T < \infty$ and $R, Q_1$and Q_2 are positive constants, $\|.\| = \max_{t\in[0,T]}|.|$,

Beside (1), we also consider the system of the following integral equations:

$$u(t,u_0) = f(t) + \int_0^t \Big[G(t,s)f\Big(s,u(s,u_0), \int_{-\infty}^{s} g(s,\tau,u(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u(\tau,u_0))d\tau\Big) - \Delta\Big]\,ds \ \dots \quad (10)$$

Where $\Delta = (\Delta_1, \Delta_2, \cdots, \Delta_n)$ is a vector parameter.

By using the following initial conditions:

$$\left.\begin{array}{l} u(0) = u(T)\ , \\ f(0) = f(T)\ . \end{array}\right\} \quad \cdots (11)$$

$$u(0) = f(0)$$

$$u(T, u_0) = f(T) + \int_0^T [G(t,s) f(s, u(s,u_0)), \int_{-\infty}^{s} g(s,\tau,u(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u(\tau,u_0))d\tau) - \Delta]\, ds \quad \cdots (12)$$

From (11), we get:

$$\Delta(t, u_0) = \frac{1}{T}\int_0^T G(t,s) f(s, u(s,u_0)), \int_{-\infty}^{s} g(s,\tau,u(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u(\tau,u_0))d\tau)\, ds \quad \cdots (13)$$

$$\Delta(t, u_0) = \lim_{m\to\infty} \Delta_m(t, u_0)$$

$$= \lim_{m\to\infty} \frac{1}{T}\int_0^T G(t,s) f(s, u_m(s,u_0)), \int_{-\infty}^{s} g(s,\tau,u_m(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u_m(\tau,u_0))d\tau)\, ds$$

We define a non-empty sets:

$$\left.\begin{array}{l} D_f = D - \frac{T}{2}MR\ , \\ D_{1f} = D_1 - \frac{T}{2}MRQ_1\ , \\ D_{2f} = D_2 - \frac{T}{2}MRQ_2\ . \end{array}\right\} \quad \cdots (14)$$

Moreover, we suppose that the greatest value of the following equation

$\Lambda = \frac{T}{2}R[L_1 + L_2Q_1 + L_3Q_2]$, does not exceed unity, i. e.

$$\Lambda < 1 \quad \cdots (15)$$

By using Lemma 3.1[5], we can state and proof the following:

Lemma 1: *Let* $f(t,u,v,w)$ *be a vector function which is defined in the interval* $0 \le t \le T$ *then:*

$$\|L(t,u_0)\| \le \alpha(t)MR \qquad \cdots (16)$$

Where $\alpha(t) = 2t(1-\frac{t}{T})$ and

$$L(t,u_0) = \int_0^t [G(t,s)f(s,u(s,u_0), \int_{-\infty}^{s} g(s,\tau,u(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u(\tau,u_0))d\tau)$$

$$- \frac{1}{T}\int_0^T G(t,s)f(s,u(s,u_0), \int_{-\infty}^{s} g(s,\tau,u(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u(\tau,u_0))d\tau)ds]ds$$

Proof:

$$\|L(t,u_0)\| \le \left\| \int_0^t [G(t,s)f(s,u(s,u_0), \int_{-\infty}^{s} g(s,\tau,u(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u(\tau,u_0))d\tau) - \right.$$

$$\left. - \frac{1}{T}\int_0^T G(t,s)f(s,u(s,u_0), \int_{-\infty}^{s} g(s,\tau,u(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u(\tau,u_0))d\tau)]ds \right\|$$

$$\le (1-\frac{t}{T})\int_0^t \|G(t,s)\| \left\| f(s,u(s,u_0), \int_{-\infty}^{s} g(s,\tau,u(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u(\tau,u_0))d\tau) \right\| ds +$$

$$+ \frac{t}{T}\int_t^T \|G(t,s)\| \left\| f(s,u(s,u_0), \int_{-\infty}^{s} g(s,\tau,u(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u(\tau,u_0))d\tau) \right\| ds$$

$$\le (1-\frac{t}{T})\int_0^t RMds + \frac{t}{T}\int_t^T RMds$$

$$\le \alpha(t)RM$$

for all $t \in [0,T]$ and $u_0 \in D_f$.

II Approximation of Solution

The study of the approximation of periodic solution for integral equation (1) will be introduced by the following theorem:

Theorem 1: *Let $f(t,u,v,w), g(t,s,u)$ and $h(t,s,u), f(t)$ be vector functions which are defined and continuous on the domain (2), satisfy the inequalities and condition (3) to (9), then there exist the sequence of functions:*

$$u_{m+1}(t,u_0) = f(t) + \int_0^t [G(t,s)f(s,u_m(s,u_0), \int_{-\infty}^{s} g(s,\tau,u_m(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u_m(\tau,u_0))d\tau) - \frac{1}{T}\int_0^T G(t,s)f(s,u_m(s,u_0), \int_{-\infty}^{s} g(s,\tau,u_m(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u_m(\tau,u_0))d\tau)ds]ds \quad \cdots (18)$$

With

$$u_0(t,u_0) = f(t) \qquad ,m = 0,1,2,\cdots,$$

Periodic in t of period T, and convergent uniformly as $m \to \infty$ in the domain:

$$(t,u_0) \in [0,T] \times D_f \qquad \cdots (19)$$

To the limit function $u^0(t,u_0)$ defined in the domain (19) which is periodic in t of period T and satisfying the system of integral equations:

$$u(t,u_0) = f(t) + \int_0^t [G(t,s)f(s,u(s,u_0), \int_{-\infty}^{s} g(s,\tau,u(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u(\tau,u_0))d\tau) - \frac{1}{T}\int_0^T G(t,s)f(s,u(s,u_0), \int_{-\infty}^{s} g(s,\tau,u(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u(\tau,u_0))d\tau)ds]ds \quad \cdots (20)$$

With

$$\|u^0(t,u_0) - u_0(t.u_0)\| \le \alpha(t)MR \qquad \cdots (21)$$

$$\|u^0(t,u_0) - u_m(t.u_0)\| \le \Lambda^m(1-\Lambda)^{-1}\alpha(t)MR \qquad \cdots (22)$$

for all $m \ge 0$, $t \in [0,T]$, where E is the identity matrix and $\alpha(t) = 2t(1-\frac{t}{T})$.

Proof: Consider the sequence of functions $u_1(t,u_0)$, $u_2(t,u_0)$, $\cdots, u_m(t,u_0)$, $\cdots$, defined by the recurrences relation (18), each of these functions of the sequence (18) are defined and continuous in the domain (2) and periodic in t of period T.

Now, by using (18) and Lemma 1, when m=0, we get:

$$\|u_1(t,x_0)-u_0(t,u_0)\| = \Bigg\| f(t) + \int_0^t [G(t,s)f(s,u_0(s,u_0), \int_{-\infty}^{s} g(s,\tau,u_0(\tau,u_0))d\tau,$$

$$, \int_{a(s)}^{b(s)} h(s,\tau,u_0(\tau,u_0))d\tau) - \frac{1}{T}\int_0^T G(t,s)f(s,u_0(s,u_0), \int_{-\infty}^{s} g(s,\tau,u_0(\tau,u_0))d\tau,$$

$$, \int_{a(s)}^{b(s)} h(s,\tau,u_0(\tau,u_0))d\tau)ds]ds - f(t)\Bigg\|$$

$$\le (1-\frac{t}{T})\int_0^t \|G(t,s)\|\|f(s,u_0(s,u_0)\,, \int_{-\infty}^{s} g(s,\tau,u_0(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u_0(\tau,u_0))d\tau)\Bigg\| ds$$

$$+\frac{t}{T}\int_t^T \|G(t,s)\|\|f(s,u_0(s,u_0), \int_{-\infty}^{s} g(s,\tau,u_0(\tau,u_0))d\tau,, \int_{a(s)}^{b(s)} h(s,\tau,u_0(\tau,u_0))d\tau)\Bigg\| ds$$

$$\le \alpha(t)MR$$

And hence

$$\|u_1(t,x_0)-u_0(t,u_0)\| \le \frac{T}{2}MR \qquad \cdots (23)$$

Also from (21), we have:

$$\|v_1(t,x_0)-v_0(t,u_0)\| = \Bigg\| \int_{-\infty}^{t} g\big(t,s,u_1(s,u_0)\big)ds - \int_{-\infty}^{t} g(t,s,u_0(t,u_0))\,ds \Bigg\|$$

$$\le \int_{-\infty}^{t} \|g(t,s,u_1(s,u_0)) - g(t,s,u_0(s,u_0))\|ds$$

$$\leq \int_{-\infty}^{t} \|K(t,s)\| \|u_1(s,u_0) - u_0(s,u_0)\| \, ds$$

$\leq Q_1 \|u_1(t,u_0) - u_0(t,u_0)\|$

$\leq \frac{T}{2} Q_1 MR.$

Therefore

$$\|v_1(t,x_0) - v_0(t,u_0)\| \leq \frac{T}{2} Q_1 MR \qquad \cdots (24)$$

for all $t \in [0,T], u_0 \in D_f$ and $v_0(t,u_0) = \int_{-\infty}^{t} g(t,s,u_0(t,u_0)) \, ds \in D_{1f}$

i.e. $v_1(t,u_0) \in D_1$, when $u_0 \in D_f$.

Again from (21), we get:

$$\|w_1(t,u_0) - w_0(t,u_0)\| = \left\| \int_{a(t)}^{b(t)} h\big(t,s,u_1(s,u_0)\big) \, ds - \int_{a(t)}^{b(t)} h(t,s,u_0) ds \right\|$$

$$\leq \int_{a(t)}^{b(t)} \|H(t,s)\| \, \|u_1(s,u_0) - u_0\| ds$$

$$\leq Q_2 \frac{T}{2} MR$$

And hence

$$\|w_1(t,u_0) - w_0(t,u_0)\| \leq \frac{T}{2} Q_2 MR \qquad \cdots (25)$$

for all $t \in [0,T],\ u_0 \in D_f$ and $w_0(t,u_0) = \int_{a(t)}^{b(t)} h(t,s,u_0) \, ds \in D_{2f}$

i.e. $w_1(t,u_0) \in D_2$, when $u_0 \in D_f$.

Thus by mathematical induction, we have:

$$\|u_m(t,u_0) - u_0\| \leq MR\alpha(t) \leq \frac{T}{2} MR \qquad \cdots (26)$$

i.e. $u_m(t,u_0) \in D$ for all $t \in [0,T], u_0 \in D_f$.

Now from (26), gives:

$$\|v_m(t,u_0)-v_0(t,u_0)\| \le \frac{T}{2}Q_1MR \qquad \cdots(27)$$

And

$$\|w_m(t,u_0)-w_0(t,u_0)\| \le \frac{T}{2}Q_2MR \qquad \cdots(28)$$

for all $t\in[0,T], u_0\in D_{f,}\ v_0(t,u_0)\in D_{1f}$, and $w_0(t,u_0)$, i.e. $v_m(t,u_0)\in D_1$

and $w_m(t,u_0)\in D_2$ for all $t\in[0,T]$ and $u_0\in D_f$.

Where

$$v_m(t,u_0)=\int_{-\infty}^{t} g\big(t,s,u_m(s,u_0)\big)ds\ ,$$

and

$$w_m(t,u_0)=\int_{a(t)}^{b(t)} h(t,s,u_m(s,u_0))\,ds$$

for all $m=0,1,2,\cdots$

We claim that the sequence of functions $u_m(t,u_0)$ is uniformly convergent on the domain(19).

For $m=1$ in (18) and using Lemma 1, we find that:

$$\|u_2(t,u_0)-u_1(t,u_0)\| \le (1-\frac{t}{T})\int_0^t \|G(t,s)\| \left\| f(s,u_1(s,u_0), \int_{-\infty}^{s} g(s,\tau,u_1(\tau,u_0)),\right.$$

$$\left., \int_{a(s)}^{b(s)} h(s,\tau,u_1(\tau,u_0))\,d\tau) - f(s,u_0, \int_{-\infty}^{s} g(s,\tau,u_0)d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u_0)d\tau \right\| ds$$

$$+\frac{t}{T}\int_t^T \|G(t,s)\| \left\| f(s,u_1(s,u_0), \int_{-\infty}^{s} g(s,\tau,u_1(\tau,u_0)), \int_{a(s)}^{b(s)} h(s,\tau,u_1(\tau,u_0))\,d\tau)\right.$$

$$-f(s,u_0,\int_{-\infty}^{s} g(s,\tau,u_0)d\tau,\int_{a(s)}^{b(s)} h(s,\tau,u_0)d\tau\Bigg\|ds$$

$$\leq (1-\frac{t}{T})\int_0^t R[L_1\|u_1(s,u_0)-u_0\| + L_2(Q_1\|u_1(s,u_0)-u_0\|) + L_3(Q_2\|u_1(s,u_0)-u_0\|)ds +$$

$$+\frac{t}{T}\int_t^T R[L_1\|u_1(s,u_0)-u_0\| + L_2(Q_1\|u_1(s,u_0)-u_0\|)+L_3(Q_2\|u_1(s,u_0)-u_0\|)ds$$

$$\leq R[L_1+L_2Q_1+L_3Q_2][(1-\frac{t}{T})\int_0^t ds+\frac{t}{T}\int_t^T ds]\|u_1(s,u_0)-u_0\|$$

$$\leq R[L_1+L_2Q_1+L_3Q_2]\ \alpha(t)\|u_1(t,u_0)-u_0\|$$

$$\leq \frac{T}{2}R[L_1+L_2Q_1+L_3Q_2]\|u_1(t,u_0)-u_0\|$$

$$\leq \frac{T}{2}R[L_1+L_2Q_1+L_3Q_2]\alpha(t)MR$$

$$\leq \alpha(t)MR\frac{T}{2}R[L_1+L_2Q_1+L_3Q_2]$$

And hence

$$\|u_2(t,u_0)-u_1(t,u_0)\| \leq \Lambda\alpha(t)MR$$

Suppose that the following inequality

$$\|u_{k+1}(t,u_0)-u_k(t,u_0)\| \leq \Lambda^{\mathrm{k}}\alpha(t)MR \qquad \cdots (29)$$

is holds for some $m=k$, then we shall to prove that:

$$\|u_{k+2}(t,u_0)-u_{k+1}(t,u_0)\| \leq \Lambda^{\mathrm{k+1}}\alpha(t)MR \qquad \cdots (30)$$

from (18) and using lemma1, when $m=k+1$ and the inequality (29) we get:

$$\|u_{k+2}(t,u_0)-u_{k+1}(t,u_0)\| \leq (1-\frac{t}{T})\int_0^t R[L_1\|u_{k+1}(s,u_0)-u_k(s,u_0)\|$$

$$+L_2(Q_1\|u_{k+1}(s,u_0)-u_k(s,u_0)\|) + L_3(Q_2\|u_{k+1}(s,u_0)-u_k(s,u_0)\|)]ds$$

$$+\frac{t}{T}\int_{t}^{T} R[L_1\|u_{k+1}(s,u_0)-u_k(s,u_0)\| + L_2(Q_1\|u_{k+1}(s,u_0)-u_k(s,u_0)\|)$$

$$+L_3(Q_2\|u_{k+1}(s,u_0)-u_k(s,u_0)\|)]ds$$

$$\leq R[L_1+L_2Q_1+L_3Q_2][(1-\frac{t}{T})\int_{0}^{t}\Lambda^{\mathrm{k}}\alpha(s)MRds+\frac{t}{T}\int_{t}^{T}\Lambda^{\mathrm{k}}\alpha(s)MRds]$$

$$\leq \frac{T}{2}R[L_1+L_2Q_1+L_3Q_2]\Lambda^{\mathrm{k}}\alpha(t)MR$$

$$\leq \Lambda^{\mathrm{k+1}}\alpha(t)MR$$

So that

$$\|u_{k+2}(t,u_0)-u_{k+1}(t,u_0)\| \leq \Lambda^{\mathrm{k+1}}\alpha(t)MR$$

By mathematical induction and by (18) and (21) the following inequality is holds:

$$\|u_{m+1}(t,u_0)-u_m(t,u_0)\| \leq \Lambda^m\,\alpha(t)\,MR \qquad \cdots (31)$$

Where $\Lambda = \frac{T}{2}R[L_1+L_2Q_1+L_3Q_2]$, for all $m = 0,1,2,\cdots$

From (31) we conclude that for $k \geq 0$,

We have the following inequality:

$$\|u_{m+k}(t,u_0)-u_m(t,u_0)\| \leq \|u_{m+k}(t,u_0)-u_{m+k-1}(t,u_0)\|$$

$$+\|u_{m+k-1}(t,u_0)-x_{m+k-2}(t,u_0)\| + \cdots + \|u_{m+1}(t,u_0)-u_m(t,u_0)\|$$

$$\leq \Lambda^{m+k-1}\|u_1(t,u_0)-u_0\| + \Lambda^{m+k-2}\|u_1(t,u_0)-u_0\| + \cdots + \Lambda^m\|u_1(t,u_0)-u_0\|$$

$$\leq \Lambda^m(1+\Lambda+\Lambda^2+\cdots+\Lambda^{k-2}+\Lambda^{k-1})\|u_1(t,u_0)-u_0\|$$

Therefore

$$\|u_{m+k}(t,u_0)-u_m(t,u_0)\| \leq \Lambda^m(1-\Lambda)^{-1}\alpha(t)MR \qquad \cdots (32)$$

Where E is identity matrix, $t \in [0,T]$ and $u_0 \in D_f$.

By using the condition (15) and the inequality (32), we find that

$$\lim_{m\to\infty}\Lambda^{\mathrm{m}} = 0 \qquad \cdots (33)$$

The relation (32) and (33) prove the uniform convergence of the sequence of function (18) in the domain (19) as $m \to \infty$.

Let

$$\lim_{m\to\infty} u_m(t,u_0) = u^0(t,u_0) \qquad \cdots (34)$$

Since the sequence of functions $u_m(t,u_0)$ is periodic in t of period T, Then the limiting function $u^0(t,x_0)$ is also periodic in t of period T.

Moreover, by the hypotheses and conditions of the theorem, the inequalities (21) and (22) are satisfied for all $m \geq 0$. ■

Theorem 2: *With the hypotheses and all conditions of the theorem 1, the periodic solution of integral equation (1) is a unique on the domain (2).*

Proof:

Let $u^*(t,u_0)$ be another periodic solution of integral equation (1), i. e.

$$u^*(t,u_0) = f(t) + \int_0^t [G(t,s)f(s,u^*(s,u_0)), \int_{-\infty}^{s} g(s,\tau,u^*(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u^*(\tau,u_0))d\tau)$$

$$-\frac{1}{T}\int_0^T G(t,s)f(s,u^*(s,u_0)), \int_{-\infty}^{s} g(s,\tau,u^*(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u^*(\tau,u_0))d\tau)ds]ds$$

and then we have

$$\|u(t,u_0) - u^*(t,u_0)\| \leq (1-\frac{t}{T})\int_0^t \|G(t,s)\| \left\| f(s,u(s,u_0)), \int_{-\infty}^{s} g(s,\tau,u(\tau,u_0)), \right.$$

$$\left. , \int_{a(s)}^{b(s)} h(s,\tau,u(\tau,u_0))\, d\tau) - f(s,u^*(s,u_0)), \int_{-\infty}^{s} g(s,\tau,u^*(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u^*(\tau,u_0))d\tau \right\| ds$$

$$+\frac{t}{T}\int_t^T \|G(t,s)\| \left\| f(s,u(s,u_0)), \int_{-\infty}^{s} g(s,\tau,u(\tau,u_0)), \int_{a(s)}^{b(s)} h(s,\tau,u(\tau,u_0))\, d\tau) \right.$$

$$\left. -f(s,u^*(s,u_0)), \int_{-\infty}^{s} g(s,\tau,u^*(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,u^*(\tau,u_0))d\tau \right\| ds$$

$$\leq (1-\frac{t}{T})\int_0^t R[L_1\|u(s,u_0)-u^*(s,u_0)\|+L_2(Q_1\|u(s,u_0)-u^*(s,u_0)\|)$$

$$+L_3(Q_2\|u(s,u_0)-u^*(s,u_0)\|)ds+\frac{t}{T}\int_t^T R[L_1\|u(s,u_0)-u^*(s,u_0)\|$$

$$+L_2(Q_1\|u(s,u_0)-u^*(s,u_0)\|)+L_3(Q_2\|u(s,u_0)-u^*(s,u_0)\|)ds$$

$$\leq R[L_1+L_2Q_1+L_3Q_2][(1-\frac{t}{T})\int_0^t ds+\frac{t}{T}\int_t^T ds]\|u(s,u_0)-u^*(s,u_0)\|$$

$$\leq R[L_1+L_2Q_1+L_3Q_2]\ \alpha(t)\|u(t,u_0)-u^*(t,u_0)\|$$

$$\leq \frac{T}{2}R[L_1+L_2Q_1+L_3Q_2]\|u(t,u_0)-u^*(t,u_0)\|\ ,$$

So that

$$\|u(t,u_0)-u^*(t,u_0)\|\leq\ \Lambda\|u(t,u_0)-u^*(t,u_0)\|.$$

By iteration we find that

$$\|u(t,u_0)-u^*(t,u_0)\|\leq\Lambda^m\|u(t,u_0)-u^*(t,u_0)\|$$

But from the condition (15), we get $\Lambda^m\to 0$ when $m\to\infty$, hence we obtain that $u(t,u_0)=u^*(t,u_0)$. In other words $u(t,u_0)$ is a unique periodic solution of (1). ■

III Existence of Solution

The problem of existence of periodic solution of period T of the system (1) is uniquely connected with existence of zero of the function $\Delta(0,u_0)=\Delta$ which has the form:

$$\Delta(t,u_0)=\frac{1}{T}\int_0^T G(t,s)f(t,u^0(t,u_0),\int_{-\infty}^t g(t,s,u^0(s,u_0))ds,\int_{a(t)}^{b(t)} h(t,s,u^0(s,u_0))d\tau)\,dt\quad\cdots(35)$$

Where $u^0(t,u_0)$ is the limiting function of the sequence of functions $u_m(t,u_0)$.

$$\Delta_m(t,u_0)=\frac{1}{T}\int_0^T G(t,s)f(t,u_m(t,u_0),\int_{-\infty}^t g(t,s,u_m(s,u_0))ds,\int_{a(t)}^{b(t)} h(t,s,u_m(s,u_0))d\tau)\,dt\quad\cdots(36)$$

for all $m = 0,1,2,\cdots$

Theorem 3: *Let all assumptions and conditions of theorem 1 and 2 are satisfied, then the following inequality is satisfied:*

$$\|\Delta(0,u_0) - \Delta_m(0,u_0)\| \le \Lambda^{m+1}(E-\Lambda)^{-1}MR \qquad \cdots (37)$$

for all $m \ge 0$, $u_0 \in D_f$.

Proof: By the iteration (35) and (36) we get:

$$\|\Delta(0,u_0) - \Delta_m(0,u_0)\| \le \frac{1}{T}\int_0^T \|G(t,s)\| \Big\| f(t,u^0(t,u_0), \int_{-\infty}^{t} g(t,s,u^0(s,u_0))ds,$$

$$, \int_{a(t)}^{b(t)} h(t,s,u^0(s,u_0))d\tau)\,dt - f(t,u_m(t,u_0), \int_{-\infty}^{t} g(t,s,u_m(s,u_0))ds,$$

$$, \int_{a(t)}^{b(t)} h(t,s,u_m(s,u_0))d\tau)\Big\|\,dt$$

From the inequalities (4) to (9), we get:

$$\|\Delta(0,u_0) - \Delta_m(0,u_0)\| \le R[L_1 + L_2Q_1 + L_3Q_2]\frac{1}{T}\int_0^T \|u^0(t,u_0) - u_m(t,u_0)\|\,dt$$

$$\le R[L_1 + L_2Q_1 + L_3Q_2]\frac{T}{2}\Lambda^m(E-\Lambda)^{-1}MR$$

But $\Lambda = R[L_1 + L_2Q_1 + L_3Q_2]$, thus the above inequality can be written as:

$\|\Delta(0,u_0) - \Delta_m(0,u_0)\| \le \Lambda^{m+1}(E-\Lambda)^{-1}MR$, i. e. the inequality (37) satisfied for all $m \ge 0$. ■

Theorem 4[4]: *Let the system (1) be defined on the interval [a, b]. Suppose that for* $m \ge 0$, *the function* $\Delta_m(0,u_0)$ *defined according to formula (36) satisfies the inequalities:*

$$\left.\begin{array}{c} \min\ \Delta_m(0,u_0) \le -\vartheta_m\ , \\ a + P \le u_0 \le b - P \\ \max\ \Delta_m(0,u_0) \ge \vartheta_m\ , \\ a + P \le u_0 \le b - P \end{array}\right\} \qquad \cdots (38)$$

Then the system (1) has periodic solution $u = u(t, u_0)$ for which $u_0 \in [a + P, b - P]$, where $P = \|MR\|\frac{T}{2}$ and $\vartheta_m = \|\Lambda^{m+1}(1 - \Lambda)^{-1}MR\|$

Proof: Let u_1, u_2 be any two points in the interval $[a + P, b - P]$ such that:

$$\left.\begin{array}{c} \Delta_m(0, u_1) = \min\ \Delta_{\mathrm{m}}(0, \mathrm{u}_0)\ , \\ a + P \le u_0 \le b - P \\ \Delta_m(0, u_2) = \max\ \Delta_{\mathrm{m}}(0, \mathrm{u}_0)\ , \\ a + P \le u_0 \le b - P \end{array}\right\} \quad \cdots (39)$$

Taking in to account inequalities (37) and (38), we have:

$$\left.\begin{array}{l} \Delta(0, u_1) = \Delta_m(0, u_1) + [\Delta(0, u_1) - \Delta_m(0, u_1)] \le 0\ , \\ \Delta(0, u_2) = \Delta_m(0, u_2) + [\Delta(0, u_2) - \Delta_m(0, u_2)] \ge 0 \end{array}\right\} \quad \cdots (40)$$

It follows from the inequalities (40) and the continuity of the function $\Delta(0, u_0)$, that there exists an isolated singular point u^0 , $u^0 \in [u_1, u_2]$, such that $\Delta(0, u^0) = 0$. This means that the system (1) has a periodic solution $u = u(t, u_0)$ for which $u_0 \in [a + P, b - P]$. ■

Remark 1: Theorem 4 is proved when u_0 is a scalar singular point which should be isolated (For this remark, see [5]).

V Stability of Solution

In this section, we study the stability of a periodic solution for the integral equation (1).

Theorem 5: *If the function $\Delta(0, u_0)$ is defined by $\Delta : D_f \to R^n$, and by the equation (35), where $u^0(t, u_0)$ is a limit of the sequence function $\{u_m(t, u_0)\}_{m=0}^{\infty}$. Then the following inequalities hold:*

$$\|\Delta(0, u_0)\| \le MR \quad \cdots (41)$$

and

$$\|\Delta(0, u_0^1) - \Delta(0, u_0^2)\| \le \frac{2}{T}\Lambda(1 - \Lambda)^{-1}\|f^1(t) - f^2(t)\| \quad \cdots (42)$$

for all $u^0, u_0^1, u_0^2 \in D_f$ and E is identity matrix.

Proof: From the properties of the function $u^0(t, u_0)$ as in theorem 1, the function $\Delta(t, u_0)$ is continuous and bounded by M in the domain $R^1 \times D_f$.

By using (35), we have:

$$\|\Delta(0,u_0^1)-\Delta(0,u_0^2)\| = \left\| \frac{1}{T}\int_0^T G(t,s)f\left(t,u^0(t,u_0^1),\int_{-\infty}^{t} g\big(t,s,u^0(s,u_0^1)\big)ds,\right.\right.$$

$$\left.,\int_{a(t)}^{b(t)} h(t,s,u^0(s,u_0^1))d\tau\right)dt - \frac{1}{T}\int_0^T G(t,s)f\left(t,u^0(t,u_0^2),\int_{-\infty}^{t} g(t,s,u^0(s,u_0^2))ds,\right.$$

$$\left.\left.,\int_{a(t)}^{b(t)} h(t,s,u^0(s,u_0^2))ds\right)dt\right\|$$

$$\le \frac{1}{T}\int_0^T \|G(t,s)\|\left\|f\left(t,u^0(t,u_0^1),\int_{-\infty}^{t} g(t,s,u^0(s,u_0^1))ds,\int_{a(t)}^{b(t)} h(t,s,u^0(s,u_0^1))ds\right)\right.$$

$$\left. -f\left(t,u^0(t,u_0^2),\int_{-\infty}^{t} g(t,s,u^0(s,u_0^2))ds,\int_{a(t)}^{b(t)} h(t,s,u^0(s,u_0^2))ds\right)\right\| dt$$

From the inequalities (4) to (9), we get:

$$\|\Delta(0,u_0^1)-\Delta(0,u_0^2)\| \le R[L_1+L_2Q_1+L_3Q_2]\frac{1}{T}\int_0^T \|u^0(t,u_0^1)-u^0(t,u_0^2)\|\,dt$$

and hence

$$\|\Delta(0,u_0^1)-\Delta(0,u_0^2)\| \le \frac{2}{\mathrm{T}}\Lambda\|u^0(t,u_0^1)-u^0(t,u_0^2)\| \qquad \cdots (43)$$

Where $u^0(t,u_0^1)$, $u^0(t,u_0^1)$ are the solution of the integral equation:

$$u(t,u_0^k) = f^k(t) + \int_0^t \Big[G(t,s)f\Big(s,u(s,u_0^k),\int_{-\infty}^{s} g(s,\tau,u(\tau,u_0^k))d\tau,\int_{a(s)}^{b(s)} h(s,\tau,u(\tau,u_0^k))d\tau\Big)$$

$$-\frac{1}{T}\int_0^T G(t,s)f\Big(s,u(s,u_0^k),\int_{-\infty}^{s} g(s,\tau,u(\tau,u_0^k))d\tau,\int_{a(s)}^{b(s)} h(s,\tau,u(\tau,u_0^k))d\tau\Big)ds\Big]ds$$

$$\cdots (44)$$

With

$u_0^k(t,u_0) = f^k(t) = u_0^k$, where k=1, 2.

From (43), we get:

$$\|u^0(t,u_0^1) - u^0(t,u_0^2)\| \le \|f^1(t) - f^2(t)\| + (1 - \frac{t}{T})\int_0^t R\big[L_1\|u^0(s,u_0^1) - u^0(s,u_0^2)\|$$

$$+L_2(Q_1\|u^0(s,u_0^1) - u^0(s,u_0^2)\|) + L_3(Q_2\|u^0(s,u_0^1) - u^0(s,u_0^2)\|)]ds$$

$$+\frac{t}{T}\int_0^t R[L_1\|u^0(s,u_0^1) - u^0(s,u_0^2)\| + L_2(Q_1\|u^0(s,u_0^1) - u^0(s,u_0^2)\|) +$$

$$+L_3(Q_2\|u^0(s,u_0^1) - u^0(s,u_0^2)\|)]ds$$

$$\le \|f^1(t) - f^2(t)\| + R[L_1 + L_2Q_1 + L_3Q_2]\alpha(t)\|u^0(t,u_0^1) - u^0(t,u_0^2)\|$$

$$\le \|f^1(t) - f^2(t)\| + \Lambda\|u^0(t,u_0^1) - u^0(t,u_0^2)\|$$

So that:

$$\|u^0(t,u_0^1) - u^0(t,u_0^2)\| \le (1-\Lambda)^{-1}\|f^1(t) - f^2(t)\| \qquad \cdots (45)$$

For all $t \in [0,T]$, $u_0^1, u_0^1 \in D_f$.

So, substituting inequality (45) in the inequality (43), we get the inequality (42). ■

Remark 2: Theorem 5, confirms the stability of the solution for the system (1), that is when a slight change happens in the point u_0, then a slight change will happen in the function $\Delta(0,u_0)$. For this remark see[4].

VI Banach Fixed Point Theorem

In this section we study the existence and uniqueness periodic of integral equation (1) will be introduced by the following:

Lemma 2[1] *Let S be a space of all continuous function on* R^1 *, for any* $z \in s$ *define* $\|z\|$ *by* $\|z\| = max_{t\in[0,T]}|z(t)|$. *Then* $(s, \|z\|)$ *is a Banach space.*

Theorem 6[1] (Banach Fixed Point Theorem)

Let E be a Banach space. If T^* *is a contraction mapping on E Then* T^* *has one and only one fixed point in E.*

Theorem 7: (Existence and Uniqueness Theorem)

Let $f(t,u,v,w), g(t,s,u), h(t,s,u)$ *and* $f(t)$ *be vectors functions which are defined and continuous and periodic in t of period T on the domain (2) and satisfying all inequalities and conditions of the theorem 1 and 2.*

Then the integral equation (1) has a unique periodic continuous solution $z(t,u_0)$ *on the domain (2), provided that* $\Lambda = \frac{T}{2}R[L_1 + L_2Q_1 + L_3Q_2]$.

Proof: Let $(C(G), \|.\|)$ is a Banach space, where $G = \{(t,u,v,w); t \in R^1, u \in D, v \in D_1, w \in D_2\}$,

Define a mapping T^* on G by

$$T^*z(t,u_0) = f(t) + \int_0^t [G(t,s)f(s,z(s,u_0), \int_{-\infty}^{s} g(s,\tau,z(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,z(\tau,u_0))d\tau)$$

$$-\frac{1}{T}\int_0^T G(t,s)f(s,z(s,u_0), \int_{-\infty}^{s} g(s,\tau,z(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,z(\tau,u_0))d\tau)ds]ds$$

Since $g(t,s,z), h(t,s,z)$, are continuous on the domain (2), then

$$\int_{-\infty}^{t} g(t,s,z(s,u_0))\,ds, \int_{a(t)}^{b(t)} h(t,s,z(s,u_0))ds$$

are also continuous on the domain (2), and $G(t,s)$ and $f(t)$ are continuous on the same domain,

So that:

$$G(t,s)f(t,z(t,u_0), \int_{-\infty\infty}^{t} g(t,s,z(s,u_0))\,ds, \int_{a(t)}^{b(t)} h(t,s,z(s,u_0))ds)$$

is continuous on the domain (2).

Thus $T^*z(t,u_0)$ is continuous on the same domain,

Hence

$T^*z(t,u_0): G \to G$

Next we claim that $T^*z(t,u_0)$ is a contraction mapping on G, let $z, w \in G$, Then

$\|T^*z(t,u_0) - T^*w(t,u_0)\| = \max_{t\in[0,T]}\{|T^*z(t,u_0) - T^*w(t,u_0)|\}$

$$= \max_{t\in[0,T]}\left\{\left|f(t)+\int_0^t [G(t,s)f(s,z(s,u_0),\int_{-\infty}^{s} g(s,\tau,z(\tau,u_0))d\tau,\int_{a(s)}^{b(s)} h(s,\tau,z(\tau,u_0))d\tau)\right.\right.$$

$$-\frac{1}{T}\int_0^T G(t,s)f(s,z(s,u_0),\int_{-\infty}^{s} g(s,\tau,z(\tau,u_0))d\tau,\int_{a(s)}^{b(s)} h(s,\tau,z(\tau,u_0))d\tau)ds]ds$$

$$-f(t)-\int_0^t [G(t,s)f(s,w(s,u_0),\int_{-\infty}^{s} g(s,\tau,w(\tau,u_0))d\tau,\int_{a(s)}^{b(s)} h(s,\tau,w(\tau,u_0))d\tau)$$

$$\left.\left.-\frac{1}{T}\int_0^T G(t,s)f(s,w(s,u_0),\int_{-\infty}^{s} g(s,\tau,w(\tau,u_0))d\tau,\int_{a(s)}^{b(s)} h(s,\tau,w(\tau,u_0))d\tau)ds]ds\right|\right\}$$

$$\leq \max_{t\in[0,T]}\left\{(1-\frac{t}{T})\int_0^t |G(t,s)|\,\left|f(s,z(s,u_0),\int_{-\infty}^{s} g(s,\tau,z(\tau,u_0))d\tau,\int_{a(s)}^{b(s)} h(s,\tau,z(\tau,u_0))d\tau)\right.\right.$$

$$\left.-f(s,w(s,u_0),\int_{-\infty}^{s} g\big(s,\tau,w(\tau,u_0)\big)d\tau,\int_{a(s)}^{b(s)} h(s,\tau,w(\tau,u_0))d\tau)\right|ds$$

$$-\frac{t}{T}\int_t^T |G(t,s)|\,\left|f(s,z(s,u_0),\int_{-\infty}^{s} g(s,\tau,z(\tau,u_0))d\tau,\int_{a(s)}^{b(s)} h(s,\tau,z(\tau,u_0))d\tau)\right.$$

$$\left.\left.-f(s,w(s,u_0)\ ,\int_{-\infty}^{s} g\big(s,\tau,w(\tau,u_0)\big)d\tau,\int_{a(s)}^{b(s)} h(s,\tau,w(\tau,u_0))d\tau)\right|ds\right\}$$

$$\leq \max_{t\in[0,T]}\left\{(1-\frac{t}{T})\int_0^t R[L_1|z(s,u_0)-w(s,u_0)|+L_2(Q_1|z(s,u_0)-w(s,u_0)|)\right.$$

$$+L_3(Q_2|z(s,u_0)-w(s,u_0)|)ds+\frac{t}{T}\int_t^T R[L_1|z(s,u_0)-w(s,u_0)|+$$

$$\left.+L_2(Q_1|z(s,u_0)-w(s,u_0)|)+L_3(Q_2|z(s,u_0)-w(s,u_0)|)ds\right\}$$

$$\leq \|R[L_1+L_2Q_1+Q_2L_3]\|\alpha(t)\max_{t\in[0,T]}\{|z(t,u_0)-w(t,u_0)|\}$$

So that

$\|T^*z(t,u_0) - T^*w(t,x_0)\| \le \Lambda\|z(t,u_0) - w(t,x_0)\|.$

Since $0 < \Lambda < 1$, we find T^* is a contraction mapping on $[0,T]$, then by theorem 6, T^* has a unique fixed point $z(t,x_0) \in [0,T]$, i. e.

$T^*z(t,x_0) = z(t,x_0)$

And

$$z(t,u_0) = f(t) + \int_0^t [G(t,s)f(s,z(s,u_0), \int_{-\infty}^{s} g(s,\tau,z(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,z(\tau,u_0))d\tau)$$

$$-\frac{1}{T}\int_0^T G(t,s)f(s,z(s,u_0), \int_{-\infty}^{s} g(s,\tau,z(\tau,u_0))d\tau, \int_{a(s)}^{b(s)} h(s,\tau,z(\tau,u_0))d\tau)ds]ds,$$

Hence $z(t,u_0)$ is the unique continuous solution for the integral equation (1) on the domain (2).

References

[1] R.N. Butris, Solution for the volterra integral equations of second kind, *M.Sc. Thesis*, (1984), Mosul University, Iraq.

[2] R.N. Butris and A. Sh. Rafeq, Existence and uniqueness solution for non-linear volterra integral equation, *J. Pure and Eng. Sciences*, 1(14) (2011), 25-29, Dohok University, Iraq.

[3] B.S. Faris, Periodic solutions for some classes of nonlinear systems of integral equations, *M. Sc. Thesis*, (2013), University of Zakho, Iraq.

[4] Yu. A. Mitropolsky and D.I. Martynyuk, *For Periodic Solutions for the Oscillations Systems with Retarded Argument*, Kiev, Ukraine, (1979).

[5] A.M. Samoilenko and N.I. Ronto, *A Numerical-Analytic Method for Investigations of Periodic Solutions*, Kiev, Ukraine, (1976).

[6] F.G. Tricomi, *Integral Equations (Third Printing)*, Turin University, Turin, Italy, (1965).